Observer dependent background geometries
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Motivation

- Metric geometry of spacetime serves multiple roles:
  - Causality
  - Observers, observables and observations
  - Gravity

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  - Loop quantum gravity
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  - Finsler spacetimes
  - Cartan geometry on observer space

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- How to serve the same roles as metric geometry?

Geometrical structures

Metric geometry

Manifold M

Lorentzian metric g

Orientation

Time orientation

Finsler geometry

Tangent bundle TM

Geometry function $L : TM \to \mathbb{R}$

Finsler function $F : TM \to \mathbb{R}$

Finsler metric $g^F(x, y)$

Cartan non-linear connection $N^a{}_b$

Cartan linear connection ∇

Cartan geometry

Lie group $G = \mathrm{ISO}_0(3, 1)$

Closed subgroup $K = \mathrm{SO}(3)$

Principal K-bundle $\pi : P \to O$

Cartan connection $A \in \Omega^1(P, \mathfrak{g})$

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From metric to Finsler

Coordinates (x^a) on M

Coordinates (x^a, y^a) on TM

Define $L(x, y) = g_{ab}(x)y^a y^b$

From Finsler to Cartan

Space O of observer 4-velocities

Space P of observer frames

Define A from connection ∇

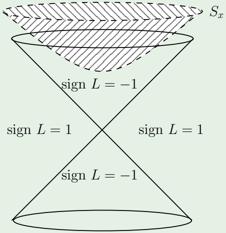
Causal structure

Metric geometry

Geometry function:

$$L = g_{ab} y^a y^b$$

y^a timelike for $L < 0$.



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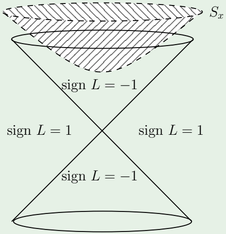


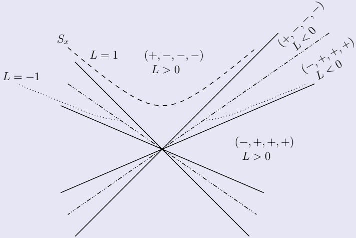
Finsler geometry

Fundamental geometry function L

Hessian:

$$g^L_{ab}(x, y) = \frac{1}{2} \bar{\partial}_a \bar{\partial}_b L(x, y)$$

Use sign of L and signature of g^L.



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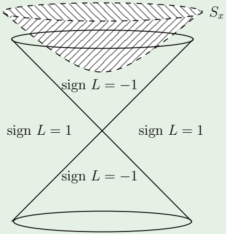

Cartan geometry

Observer space:

$$O = \bigcup_{x \in M} S_x$$

O contains only future unit timelike vectors.

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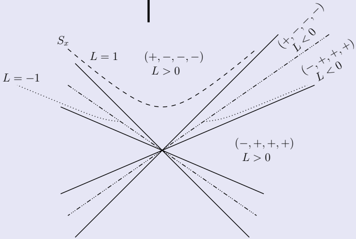

Observers

Metric geometry

Timelike curve γ:

$$\begin{array}{rcccc} \gamma & : & \mathbb{R} & \to & M \\ & & \tau & \mapsto & \gamma(\tau) \end{array}$$

$$g_{ab}\dot{\gamma}^a\dot{\gamma}^b = -1$$

Orthonormal frame f:

$$f_0^a = \dot{\gamma}^a$$

$$g_{ab}f_i^a f_j^b = \eta_{ij}$$

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$$\begin{array}{cccc} \gamma & : & \mathbb{R} & \rightarrow & M \\ & & \tau & \mapsto & \gamma(\tau) \end{array}$$

$$\dot{\gamma}(\tau) \in S_{\gamma(\tau)} \subset TM$$

Canonical lift Γ:

$$\Gamma(\tau) = (\gamma(\tau), \dot{\gamma}(\tau))$$

$$\Gamma(\tau) \in O \subset TM$$

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Cartan geometry

Observer curve Γ:

$$\begin{array}{cccccc} \Gamma & : & \mathbb{R} & \to & O \\ & & \tau & \mapsto & \Gamma(\tau) \end{array}$$

Lift condition:

$$\tilde{e}^i\dot{\Gamma}(\tau) = \delta^i_0$$

Orthonormal frame f:

$$f \in \pi^{-1}(\Gamma(\tau)) \subset P$$

Inertial observers

Metric geometry

Minimize arc length integral:

$$\int_{t_1}^{t_2} \sqrt{|g_{ab}(\gamma(t))\dot{\gamma}^a(t)\dot{\gamma}^b(t)|}dt$$

Geodesic equation:

$$\ddot{\gamma}^a + \Gamma^a{}_{bc}\dot{\gamma}^b\dot{\gamma}^c = 0$$

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Finsler geometry

Minimize arc length integral:

$$\int_{t_1}^{t_2} F(\gamma(t), \dot{\gamma}(t))dt$$

Geodesic equation:

$$\ddot{\gamma}^a + N^a{}_b\dot{\gamma}^b = 0$$

Geodesic spray:

$$\mathbf{S} = y^a(\partial_a - N^b{}_a\bar{\partial}_b)$$

Integral curves:

$$\dot{\Gamma}(\tau) = \mathbf{S}(\Gamma(\tau))$$

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Cartan geometry

Geodesic condition:

$$\tilde{b}^\alpha \dot{\Gamma}(\tau) = 0$$

Integral curves:

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Field theory

Metric geometry

Tensor bundle:

$$T^{r,s}M = TM^{\otimes r} \otimes T^*M^{\otimes s}$$

Tensor fields Φ in $T^{r,s}M$

Observer frame f

Tensor components of Φ wrt f

Field theory

Metric geometry

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Tensor components of Φ wrt f

Finsler geometry

Horizontal tensor bundle:

$$H^{r,s}TM = HTM^{\otimes r} \otimes H^*TM^{\otimes s}$$

Horizontal tensor fields Φ in $H^{r,s}TM$

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Tensor components of Φ wrt f

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Horizontal tensor fields Φ in $H^{r,s}TM$

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Horizontal tensor bundle:

$$H^{r,s}O = HO^{\otimes r} \otimes H^*O^{\otimes s}$$

Horizontal tensor fields Φ in $H^{r,s}O$

Horizontal basis $\underline{\tilde{e}}_i$

Tensor components of Φ wrt $\underline{\tilde{e}}_i$

Gravity

Metric geometry

Einstein-Hilbert action: $$S_{\text{EH}} = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\,R$$

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Using non-linear connection:

$$S_{\mathrm{N}} = \frac{1}{\kappa}\int_\Sigma \mathrm{Vol}_{\tilde{G}}\,R^a{}_{ab}y^b$$

Using linear connection:

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Using horizontal vector fields:

$$S_{\mathrm{H}} = \int_O \tilde{b}^\alpha([\underline{\tilde{e}}_\alpha, \underline{\tilde{e}}_0])\,\mathrm{Vol}_O$$

Using Cartan curvature:

$$S_{\mathrm{C}} = \int_O \kappa_{\mathfrak{h}}(\tilde{F}_{\mathfrak{h}} \wedge \tilde{F}_{\mathfrak{h}}) \wedge \mathrm{Vol}_S$$

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Spacetime

Metric geometry

Spacetime manifold M

Finsler geometry

Spacetime manifold M

Spacetime

Metric geometry

Spacetime manifold M

Finsler geometry

Spacetime manifold M

Cartan geometry

Cartan connection $A \in \Omega^1(P, \mathfrak{g})$ induces split of TP:

$$\begin{array}{ccccccccc}
R_pP & \oplus & B_pP & \oplus & \vec{H}_pP & \oplus & H^0_pP & = & T_pP \\
\Omega \downarrow & + & b \downarrow & + & \vec{e} \downarrow & + & e^0 \downarrow & = & A \downarrow \\
\mathfrak{k} & \oplus & \mathfrak{y} & \oplus & \vec{\mathfrak{z}} & \oplus & \mathfrak{z}^0 & = & \mathfrak{g}
\end{array}$$

If $RP \oplus BP$ is integrable:

$\Rightarrow$ Foliation $\mathcal{F}$ exists.

$\Rightarrow$ Spacetime M as leaf space.

If $RP \oplus BP$ is non-integrable:

$\Rightarrow$ No foliation exists.

$\Rightarrow$ No underlying spacetime.

Causality, observers, observables and gravity defined in both cases.

Summary

- Finsler spacetimes
  - Generalization of metric spacetimes.
  - Geometry defined by function L on TM.
  - Lengths measured by Finsler function $F = |L|^{\frac{1}{n}}$.
  - Metric generalized by Finsler metric g^F_{ab}.

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- Different geometries provide compatible definitions of:
  - Causality
  - Observers
  - Observables
  - Gravity

Open questions

- Experimental effects of non-tensorial structures?
- Properties of matter (gauge) theories on these backgrounds?
- Quantization of these structures?

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