Fluid dynamics with cosmological symmetry on Finsler spacetimes

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Outline

Motivation

- So far unexplained cosmological observations:
  - Accelerating expansion of the universe
  - Homogeneity of cosmic microwave background

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- Idea here: modification of the geometric structure of spacetime!
  - Replace metric spacetime geometry by Finsler geometry.
  - Similarly: replacing flat spacetime by curved spacetime led to GR.

Fluids are everywhere

- Perfect fluid:
  - No shear stress, no friction.
  - Characterized by density ρ and pressure p.
    - Dust, dark matter: $p = 0$.
    - Radiation: $p = \frac{1}{3}\rho$.
    - Dark energy: $p < -\frac{1}{3}\rho$.
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- Charged, multi-component gas:
  - Plasma, interacting gas including recombination / ionization.
  - Used in stellar dynamics, pre-CMB era models...

From spacetime to observer space

- Fluid dynamics naturally lift to tangent bundle:
  - Fluids conveniently modeled by particle dynamics (SPH. . .).
  - Physical fluids constituted by particles.
  - Particle trajectories lift to tangent bundle: $\gamma \rightsquigarrow (\gamma, \dot{\gamma})$.

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  - Physical observer velocities are future unit timelike vectors.

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- Space of observers corresponds to particle tangent vectors.

$\Rightarrow$ Consider fluid dynamics on observer space!

Finsler spacetimes

- Finsler geometry of space widely used in physics:
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  - Electrodynamics in anisotropic media
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  - Gravity
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  - Other matter field theories
- Possible explanations of yet unexplained phenomena:
  - Fly-by anomaly
  - Galaxy rotation curves
  - Accelerating expansion of the universe

Outline

The clock postulate

- Proper time along a curve in Lorentzian spacetime:

$$\tau = \int_{t_1}^{t_2} \sqrt{-g_{ab}(x(t))\dot{x}^a(t)\dot{x}^b(t)}dt\,.$$

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$$\tau = \int_{t_1}^{t_2} \sqrt{-g_{ab}(x(t))\dot{x}^a(t)\dot{x}^b(t)}dt\,.$$

- Finsler geometry: use a more general length functional:

$$\tau = \int_{t_1}^{t_2} F(x(t), \dot{x}(t))dt\,.$$

- Finsler function $F : TM \to \mathbb{R}^+$.
- Parametrization invariance requires homogeneity:

$$F(x, \lambda y) = \lambda F(x, y) \quad \forall \lambda > 0\,.$$

Finsler spacetimes

- Finsler geometries suitable for spacetimes exist. [C. Pfeifer, M. Wohlfarth '11]

$\Rightarrow$ Finsler metric with Lorentz signature:

$$g^F_{ab}(x, y) = \frac{1}{2}\frac{\partial}{\partial y^a}\frac{\partial}{\partial y^b}F^2(x, y)\,.$$

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- Unit vectors $y \in T_xM$ defined by

$$F^2(x,y) = g^F_{ab}(x,y)y^a y^b = 1\,.$$

⇒ Set $\Omega_x \subset T_xM$ of unit timelike vectors at $x \in M$.

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- Ω_x contains a closed connected component $S_x \subseteq \Omega_x$.

$\rightsquigarrow$ Causality: S_x corresponds to physical observers.

Geometry on the tangent bundle

- Cartan non-linear connection:

$$N^a{}_b = \frac{1}{4}\bar{\partial}_b \left[g^{F\,ac}(y^d\partial_d\bar{\partial}_c F^2 - \partial_c F^2)\right]$$

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$\Rightarrow$ Split of the tangent and cotangent bundles:

- Tangent bundle: $TTM = HTM \oplus VTM$

$$\delta_a = \partial_a - N^b{}_a \bar{\partial}_b \,, \quad \bar{\partial}_a$$

- Cotangent bundle: $T^*TM = H^*TM \oplus V^*TM$

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$$G = -g^F_{ab}\,dx^a \otimes dx^b - \frac{g^F_{ab}}{F^2}\,\delta y^a \otimes \delta y^b$$

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- Geodesic spray:

$$\mathbf{S} = y^a\delta_a$$

Finsler geometry in general coordinates

- Encode induced coordinates using natural tensor fields on TM:
  - Tangent structure $J = \bar{\partial}_a \otimes dx^a$.
  - Cotangent structure $J^* = dx^a \otimes \bar{\partial}_a$.
  - Liouville vector field $\mathbf{c} = y^a \bar{\partial}_a$.
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- Write tensor fields in arbitrary coordinates om TM.
- Write Finsler geometry using these tensor fields:
  - Homogeneity of Finsler function: $\mathbf{c}F = F$.
  - Cartan one-form $\Theta = \frac{1}{2} J^* \left(dF^2\right)$.
  - Cartan two-form $\Omega = d\Theta$ is symplectic.
  - Geodesic spray: unique vector field $\mathbf{s}$ such that

$$\iota_{\mathbf{s}} \Omega = -dF^2/2\,.$$

  - Projectors h on HTM and v on VTM.
  - Adjoint structure: unique $(1, 1)$ tensor field θ such that

$$\theta \circ h = v \circ \theta = 0\,, \quad \theta \circ J = v\,, \quad J \circ \theta = h\,.$$

  - Sasaki metric:

$$G(X, Y) = \Omega(X, J(Y)) - \Omega(X, \theta(Y))/F^2\,.$$

Geometry on observer space

- Recall from the definition of Finsler spacetimes:
  - Set $\Omega_x \subset T_xM$ of unit timelike vectors at $x \in M$.
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- Definition of observer space:

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- Geodesic hypersurface measure $\omega = \iota_{\mathbf{r}}\Sigma$.
- Note that $\mathcal{L}_{\mathbf{r}}\Sigma = 0$ and $d\omega = 0$.

From metric to Finsler geometry

Tangent bundle geometry:

- Finsler function:

$$F(x, y) = \sqrt{|g_{ab}(x)y^a y^b|}$$

- Finsler metric:

$$g^F_{ab}(x, y) = \begin{cases} -g_{ab}(x) & y \text{ timelike} \\ g_{ab}(x) & y \text{ spacelike} \end{cases}$$

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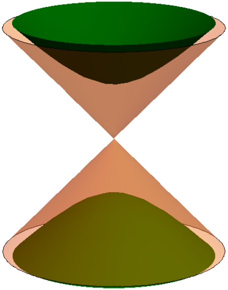
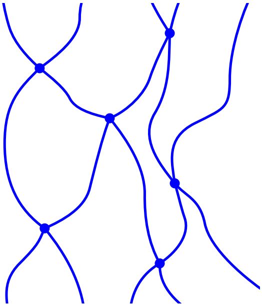


- Observer space:
  - Space Ω_x of unit timelike vectors at $x \in M$.
  - Space S_x of future unit timelike vectors at $x \in M$.
  - Observer space O: union of shells S_x.

Outline

Definition of fluids

- Single-component fluid:
  - Constituted by classical, relativistic particles.
  - Particles have equal properties (mass, charge, . . .).
  - Particles follow piecewise geodesic curves.
  - Endpoints of geodesics are interactions with other particles.



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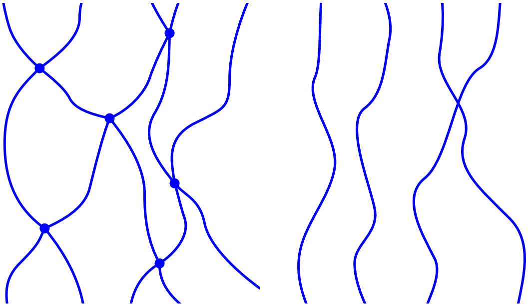

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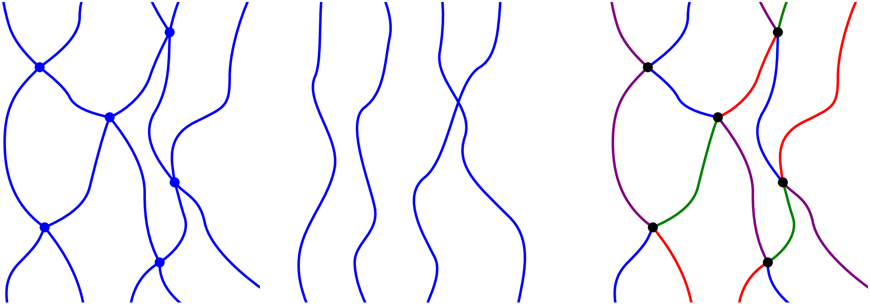
- Multi-component fluid: multiple types of particles.



Geodesics on observer space

- Dynamics of fluids depends on geodesic equation.
- Geodesic equation for curve $x(\tau)$ on spacetime M:

$$\ddot{x}^a + N^a{}_b(x, \dot{x})\dot{x}^b = 0\,.$$

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- Canonical lift of curve to tangent bundle TM:

$$x\,, \quad y = \dot{x}\,.$$

- Lift of geodesic equation:

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- Tangent vectors are future unit timelike: $(x, y) \in O$.

⇒ Particle trajectories are piecewise integral curves of **r** on O.

One-particle distribution function

- Recall: $\omega = \iota_{\mathbf{r}}\Sigma \in \Omega^6(O)$ unique 6-form such that:
  - ω non-degenerate on every hypersurface not tangent to $\mathbf{r}$.
  - $d\omega = 0$.

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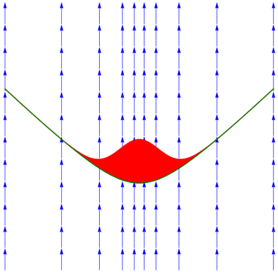
- Define one-particle distribution function $\phi : O \to \mathbb{R}^+$ such that:

For every hypersurface $\sigma \subset O$,

$$N[\sigma] = \int_\sigma \phi\omega$$

# of particle trajectories through σ.



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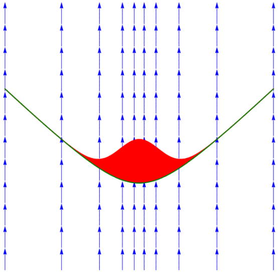

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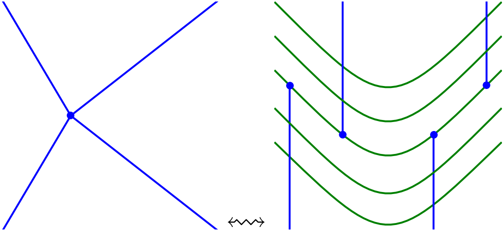
  - Counting of particle trajectories respects hypersurface orientation.
- For multi-component fluids: ϕ_i for each component i.

Collisions & the Liouville equation

- Collision in spacetime $\leftrightsquigarrow$ interruption in observer space.



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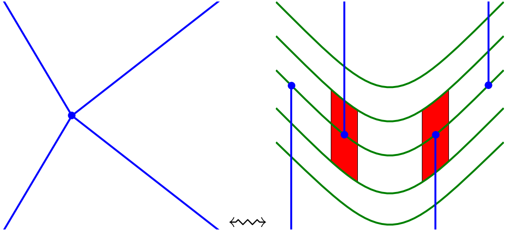

- For any open set $V \in O$,

$$\int_{\partial V} \phi\omega = \int_V d(\phi\omega) = \int_V \mathcal{L}_{\mathbf{r}}\phi\Sigma$$

# of outbound trajectories - # of inbound trajectories.

$\Rightarrow$ Collision density measured by $\mathcal{L}_{\mathbf{r}}\phi$.

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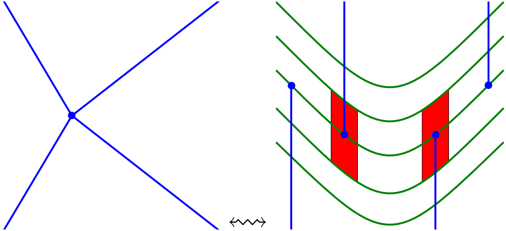

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$\Rightarrow$ Collision density measured by $\mathcal{L}_{\mathbf{r}}\phi$.

- Collisionless fluid: trajectories have no endpoints, $\mathcal{L}_{\mathbf{r}}\phi = 0$.

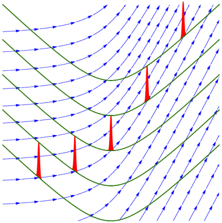
$\Rightarrow$ Simple, first order equation of motion for collisionless fluid.

$\Rightarrow$ ϕ is constant along integral curves of $\mathbf{r}$.

Examples of fluids

Geodesic dust fluid:

$$\phi(x, y) \sim \delta(y - u(x)) \,.$$



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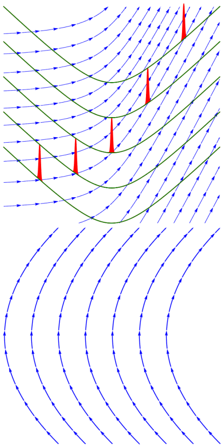

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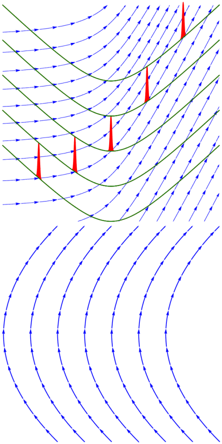

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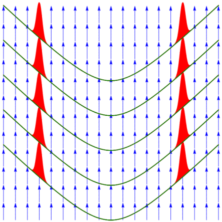
Collisionless fluid:

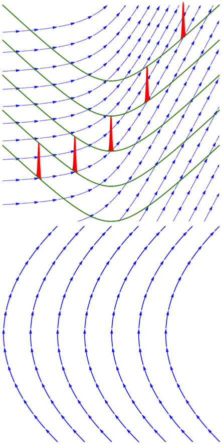
$$\mathcal{L}_{\mathbf{r}}\phi = 0\,.$$



Examples of fluids

Geodesic dust fluid:

$$\phi(x, y) \sim \delta(y - u(x)) \,.$$



"Jenkka"

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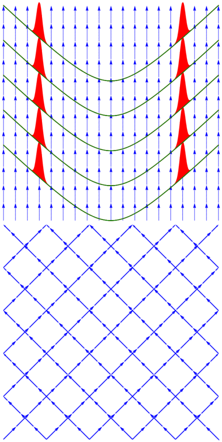

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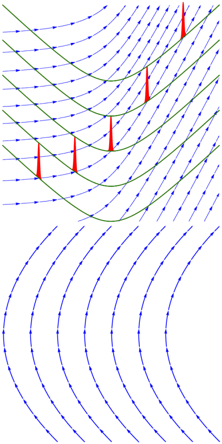

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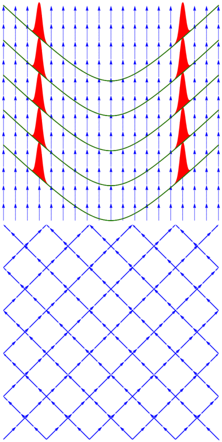

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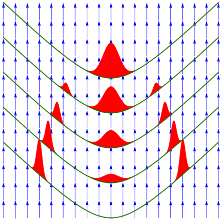

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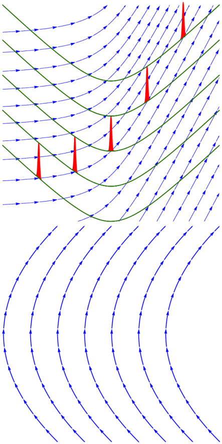

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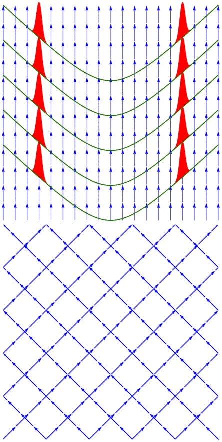

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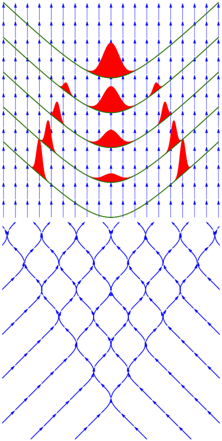

“Humppa”

Outline

Symmetry generating vector fields

- Spherical coordinates $(x^a) = (t, r, \theta, \varphi)$.
- Generators of cosmological symmetry:

$$\xi_1 = \sqrt{1 - kr^2}\left(\sin\theta\cos\varphi\partial_r + \frac{\cos\theta\cos\varphi}{r}\partial_\theta - \frac{\sin\varphi}{r\sin\theta}\partial_\varphi\right),$$

$$\xi_2 = \sqrt{1 - kr^2}\left(\sin\theta\sin\varphi\partial_r + \frac{\cos\theta\sin\varphi}{r}\partial_\theta + \frac{\cos\varphi}{r\sin\theta}\partial_\varphi\right),$$

$$\xi_3 = \sqrt{1 - kr^2}\left(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta\right),$$

$$\xi_4 = \sin\varphi\partial_\theta + \frac{\cos\varphi}{\tan\theta}\partial_\varphi,$$

$$\xi_5 = -\cos\varphi\partial_\theta + \frac{\sin\varphi}{\tan\theta}\partial_\varphi,$$

$$\xi_6 = \partial_\varphi.$$

- ↯ Canonical lifts to TM are lengthy in induced coordinates (x^a, y^a).

Non-induced tangent bundle coordinates

- Non-induced coordinates $(Z^A) = (\hat{t}, \hat{r}, \hat{\theta}, \hat{\varphi}, \hat{y}, \hat{u}, \hat{v}, \hat{w})$ on TM:

$$t = \hat{t}\,, \quad r = \hat{r}\,, \quad \theta = \hat{\theta}\,, \quad \varphi = \hat{\varphi}\,, \quad y^\theta = \frac{\hat{w}}{\hat{r}} \sin\hat{u}\cos\hat{v}\,,$$

$$y^t = \hat{y}\,, \quad y^r = \hat{w}\cos\hat{u}\sqrt{1 - k\hat{r}^2}\,, \quad y^\varphi = \frac{\hat{w}}{\hat{r}\sin\hat{\theta}} \sin\hat{u}\sin\hat{v}\,.$$

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$\Rightarrow$ Canonical lifts of cosmological symmetry generators simplify.

$\Rightarrow$ Most general cosmologically symmetric function $f : TM \to \mathbb{R}$:

$$f = f(\hat{t}, \hat{w}, \hat{y})\,.$$

$\Rightarrow$ Describe Finsler geometry using these non-induced coordinates.

Tangent bundle geometry

- Tangent structure:

$$J^A{}_B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sin\hat{u}}{\hat{w}\sqrt{1-k\hat{r}^2}} & \frac{\hat{r}\cos\hat{u}\cos\hat{v}}{\hat{w}} & \frac{\hat{r}\cos\hat{u}\sin\hat{v}\sin\hat{\theta}}{\hat{w}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\hat{r}\sin\hat{v}}{\hat{w}\sin\hat{u}} & \frac{\hat{r}\cos\hat{v}\sin\hat{\theta}}{\hat{w}\sin\hat{u}} & 0 & 0 & 0 & 0 \\ 0 & \frac{\cos\hat{u}}{\sqrt{1-k\hat{r}^2}} & \hat{r}\sin\hat{u}\cos\hat{v} & \hat{r}\sin\hat{u}\sin\hat{v}\sin\hat{\theta} & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Cotangent structure: $J^*_A{}^B = J^B{}_A$.
- Liouville vector field: $\mathbf{c} = \hat{y}\hat{\partial}_y + \hat{w}\hat{\partial}_w$.

Finsler geometry and geodesic spray

- Most general cosmologically symmetric Finsler function:

$$F = F(\hat{t}, \hat{y}, \hat{w})\,.$$

- Homogeneity condition:

$$\mathbf{c}F = F \quad \Rightarrow \quad F = \hat{y}\tilde{F}(\hat{t}, \hat{w}/\hat{y})\,.$$

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⇒ Geodesic spray:

$$\begin{aligned}\mathbf{S} &= \hat{y}\hat{\partial}_t + \hat{w}\cos\hat{u}\sqrt{1-k\hat{r}^2}\hat{\partial}_r + \frac{\hat{w}\sin\hat{u}\cos\hat{v}}{\hat{r}}\hat{\partial}_\theta + \frac{\hat{w}\sin\hat{u}\sin\hat{v}}{\hat{r}\sin\hat{\theta}}\hat{\partial}_\varphi \\ &\quad - \frac{\hat{w}\sin\hat{u}\sqrt{1-k\hat{r}^2}}{\hat{r}}\hat{\partial}_u - \frac{\hat{w}\sin\hat{u}\sin\hat{v}}{\hat{r}\tan\hat{\theta}}\hat{\partial}_v - \hat{y}^2\frac{\tilde{F}_{ww}\tilde{F}_t - \tilde{F}_w\tilde{F}_{tw}}{\tilde{F}\tilde{F}_{ww}}\hat{\partial}_y \\ &\quad - \hat{y}\frac{\hat{w}\tilde{F}_t\tilde{F}_{ww} + \hat{y}\tilde{F}\tilde{F}_{tw} - \hat{w}\tilde{F}_w\tilde{F}_{tw}}{\tilde{F}\tilde{F}_{ww}}\hat{\partial}_w\,.\end{aligned}$$

Observer space and Reeb vector field

- Introduce new coordinates:

$$\tilde{t} = \hat{t}\,, \quad \tilde{r} = \hat{r}\,, \quad \tilde{\theta} = \hat{\theta}\,, \quad \tilde{\varphi} = \hat{\varphi}\,, \quad \tilde{u} = \hat{u}\,, \quad \tilde{v} = \hat{v}\,,$$

$$\tilde{y} = \hat{y}\tilde{F}\left(\hat{t}, \frac{\hat{w}}{\hat{y}}\right)\,, \quad \tilde{w} = \frac{\hat{w}}{\hat{y}}\,.$$

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$$F(\hat{t}, \hat{y}, \hat{w}) = \tilde{y} = \hat{y}\tilde{F}\left(\hat{t}, \frac{\hat{w}}{\hat{y}}\right)\,, \quad \tilde{w} = \frac{\hat{w}}{\hat{y}}\,.$$

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$\Rightarrow$ Reeb vector field:

$$\mathbf{r} = \frac{1}{\tilde{F}}\left(\tilde{w}\cos\tilde{u}\sqrt{1 - k\tilde{r}^2}\tilde{\partial}_r + \frac{\tilde{w}\sin\tilde{u}\cos\tilde{v}}{\tilde{r}}\tilde{\partial}_\theta + \frac{\tilde{w}\sin\tilde{u}\sin\tilde{v}}{\tilde{r}\sin\tilde{\theta}}\tilde{\partial}_\varphi \right.$$
$$\left. - \frac{\tilde{w}\sin\tilde{u}\sqrt{1 - k\tilde{r}^2}}{\tilde{r}}\tilde{\partial}_u - \frac{\tilde{w}\sin\tilde{u}\sin\tilde{v}}{\tilde{r}\tan\tilde{\theta}}\tilde{\partial}_v - \frac{\tilde{F}_{tw}}{\tilde{F}_{ww}}\tilde{\partial}_w + \tilde{\partial}_t\right)\,.$$

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- Example: collisionless dust fluid $\phi(x, y) \sim \rho(x)\delta_{S_x}(y, u(x))$:

$$u(t) = \frac{1}{\tilde{F}(t, 0)}\partial_t , \quad \partial_t\left(\rho(t)\sqrt{g^F(t, 0)}\right) = 0 .$$

Outline

Summary

- Finsler spacetimes:
  - Define geometry by length functional.
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- Cosmology:
  - Useful choice of non-induced coordinates on TM.
  - All quantities depend on only two coordinates $t, \tilde{w}$.
  - Simple equation of motion for collisionless fluid matter.

Outlook

- Derivation of Finsler-Friedmann equations:
  - Consider Finsler geometric theory of gravity.
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  - Inflation?