Observables from spherically symmetric modified dispersion relations

D. Läänemets, MH and C. Pfeifer, arXiv:2201.04694 [gr-qc]

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Center of Excellence “The Dark Side of the Universe”

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Outline

Why study modified dispersion relations?

- Observations in astronomy and cosmology rely on “messengers”:
  - Photons - wide energy range from radio to gamma.
  - Other particles (predominantly protons and neutrinos).
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  - Modified theories of gravity and extra fields.

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- MDR may in general introduce energy-dependence of these effects.

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Dispersion relations as Hamiltonians

- Hamiltonian picture of point mass dynamics:
  - Describe particle motion in position-momentum variables (x^μ, p_μ).
  - Variables are coordinates on the cotangent bundle T^*M of spacetime M.
  - Introduce abbreviations:

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \bar{\partial}^\mu = \frac{\partial}{\partial p_\mu}.$$

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- Dynamics governed by Hamiltonian $H(x, p)$:
  - Dispersion relation defines "mass shell" of point mass:

$$H(x, p) = -\frac{m^2}{2}.$$

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- Point mass Hamiltonian in general relativity:
  - Metric $g_{\mu\nu}(x)$ defines $H(x, p) = \frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu$.

  $\Rightarrow$ Equations of motion give geodesic equation.

Static spherically symmetric modified dispersion relations

- Introduce spherical position-momentum variables:

$$(x^\mu) = (t, r, \theta, \phi)\,, \quad (p_\mu) = (p_t, p_r, p_\theta, p_\phi)\,.$$

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$$H = H(r, p_t, p_r, w)\,, \quad w^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\,.$$

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- Energy $\mathcal{E} = p_t$:

$$\partial_t H = 0 \quad \Rightarrow \quad 0 = \dot{p}_t\,.$$

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⇒ Angular equations of motion solved by equatorial motion $\theta = \frac{\pi}{2}$, $p_\theta = 0$:

$$\dot{\theta} = \frac{\partial H}{\partial w}\frac{1}{w}p_\theta, \quad \dot{p}_\theta = \frac{\partial H}{\partial w}\frac{1}{w}\frac{\cos\theta}{\sin^3\theta}p_\phi^2.$$

General linear modified dispersion relation

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- General relativity in vacuum implies Schwarzschild spacetime:

$$b = a^{-1} = 1 - \frac{r_s}{r}\,.$$

κ-Poincaré dispersion relation

- General form with Planck length ℓ and vector field Z^μ satisfying $g_{\mu\nu}Z^\mu Z^\nu = -1$:

$$H(x,p) = -\frac{2}{\ell^2}\sinh\left(\frac{\ell}{2}Z^\mu(x)p_\mu\right)^2 + \frac{1}{2}e^{\ell Z^\mu(x)p_\mu}\left[g^{\mu\nu}(x)p_\mu p_\nu + (Z^\mu(x)p_\mu)^2\right].$$

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- Possible choice: $c = \sqrt{a}$, $d = 0$.

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⇒ Photon orbit radius determines “shadow” ⇒ observable signature.

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  - Background value independent of photon momentum:

$$r_0 = -2\frac{a_0}{a_0'}\,.$$

  - First order correction depends on photon momentum:

$$r_1 = \frac{2r_0^4(a_0^2\partial_r h_0 - a_0 a_0' h_0)}{\mathcal{L}_0^2\left(r_0^2 a_0 a_0'' - r_0^2 a_0'^2 - 2r_0 a_0 a_0' - 6a_0^2\right)}\,.$$

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- Consider Schwarzschild background:
  - Background value independent of photon momentum:

$$r_0 = \frac{3}{2}r_s\,.$$

  - First order correction depends on photon momentum:

$$r_1 = \frac{9r_s^3}{16\mathcal{L}_0^2}(4h_0 + 3r_s\partial_r h_0)\,.$$

κ-Poincaré dispersion relation

- Photon orbits determined from transcendental equation:

$$\frac{2\mathcal{L}}{r \pm \ell\mathcal{L}} \mp \frac{ra'}{\ell a} \ln\left(\frac{r}{r \pm \ell\mathcal{L}}\right) = 0\,.$$

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$\Rightarrow$ Momentum-dependent modification of order $\sim \ell\mathcal{L}$.

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General solution method

- Model Shapiro delay with radar experiment:
  - Signal emitted at radial coordinate $r = r_e$.
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- Total signal travel time:

$$\Delta t = \int_{r_e}^{r_c} \mathrm{d}r \left.\frac{\mathrm{d}t}{\mathrm{d}r}\right|_{\dot{r}<0} + \int_{r_c}^{r_m} \mathrm{d}r \left.\frac{\mathrm{d}t}{\mathrm{d}r}\right|_{\dot{r}>0} + \int_{r_m}^{r_c} \mathrm{d}r \left.\frac{\mathrm{d}t}{\mathrm{d}r}\right|_{\dot{r}<0} + \int_{r_c}^{r_e} \mathrm{d}r \left.\frac{\mathrm{d}t}{\mathrm{d}r}\right|_{\dot{r}>0} .$$

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- Pay attention to divergence $\dot{r} = 0$ at point $r = r_c$ of closest approach!

General linear modified dispersion relation

- Use condition $\dot{r} = 0$ to determine closest approach r_c:

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- $\dot{t}$ and $\dot{r}$ as functions of r and constant parameters:

$$\frac{dt}{dr} = -\frac{a(r)\mathcal{E}}{b(r)p_r} + \epsilon(\ldots)\,.$$

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- Use $p_r|_{r_c} = 0$ in massless dispersion relation to relate $r_c, \mathcal{E}, \mathcal{L}$:

$$0 = \mathcal{H}(r_c, \mathcal{E}, 0, \mathcal{L}) = \frac{\mathcal{L}^2}{2r_c^2} e^{\ell\mathcal{E}\sqrt{a(r_c)}} - \frac{2}{\ell^2}\sinh\left(\frac{\ell}{2}\mathcal{E}\sqrt{a(r_c)}\right)\,.$$

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Outline

General solution method

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  - Incoming light ray from "infinity" $r \to \infty$.
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- Possible to use p_r and relation between $r_c, \mathcal{E}, \mathcal{L}$ from Shapiro delay.

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