Observer space geometry
Introduction to Finsler and Cartan geometries

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Outline

1. Introduction

Motivation

- Pseudo-Riemannian geometry of spacetime has multiple roles:
  - Causality
  - Observers, observables and observations
  - Gravity

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  - Spin foam networks
  - Causal dynamical triangulations

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- How to serve the same roles as pseudo-Riemannian geometry?

Why Finsler geometry of spacetimes?

- Finsler geometry of space widely used in physics:
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- ? Possible explanations of yet unexplained phenomena:
  - ? Galaxy rotation curves
  - ? Accelerating expansion of the universe

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- Solution:
  - Consider space *O* of all allowed observers.
  - Describe experiments on observer space instead of spacetime.
  - ⇒ Observer dependence of physical quantities follows naturally.
  - ⇒ No preferred observers.
  - Geometry of observer space modeled by Cartan geometry.

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$\Rightarrow$ Arc length for curves $t \mapsto \gamma(t) \in M$ defined by the metric:

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- Observables are components of tensor fields.
- Tensor components must be expressed in suitable basis.
$\Rightarrow$ Metric provides notion of orthonormal frames:

$$g_{ab}f_i^a f_j^b = \eta_{ij}\,.$$

$\Rightarrow$ Orthogonal frame bundle $\tilde{\pi} : P \to M$ is principal $SO(1,3)$-bundle.

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Basics of Finsler spacetimes

- Finsler geometry defined by length functional for curve γ:

$$\tau_2 - \tau_1 = \int_{t_1}^{t_2} F(\gamma(t), \dot{\gamma}(t)) dt$$

- Finsler function $F : TM \to \mathbb{R}^+$.
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- Introduce manifold-induced coordinates (x^a, y^a) on TM:
  - Coordinates x^a on M.
  - Define coordinates y^a for $y^a \frac{\partial}{\partial x^a} \in T_x M$.
  - Tangent bundle TTM spanned by $\left\{ \partial_a = \frac{\partial}{\partial x^a}, \bar{\partial}_a = \frac{\partial}{\partial y^a} \right\}$.

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- n-homogeneous functions on TM: $f(x, \lambda y) = \lambda^n f(x, y)$.
  - n-homogeneous smooth geometry function $L : TM \to \mathbb{R}$.
  - $\Rightarrow$ 1-homogeneous Finsler function $F = |L|^{\frac{1}{n}}$.
- $\Rightarrow$ Finsler metric with Lorentz signature:

$$g_{ab}^F(x, y) = \frac{1}{2} \bar{\partial}_a \bar{\partial}_b F^2(x, y).$$

Connections on Finsler spacetimes

- Cartan non-linear connection:

$$N^a{}_b = \frac{1}{4}\bar{\partial}_b\left[g^{F\,ac}(y^d\partial_d\bar{\partial}_c F^2 - \partial_c F^2)\right]\,.$$

⇒ Berwald basis of TTM:

$$\{\delta_a = \partial_a - N^b{}_a\bar{\partial}_b, \bar{\partial}_a\}\,.$$

⇒ Dual Berwald basis of T^*TM:

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- Cartan linear connection:

$$\nabla_{\delta_a} \delta_b = F^c{}_{ab} \delta_c , \; \nabla_{\delta_a} \bar{\partial}_b = F^c{}_{ab} \bar{\partial}_c , \; \nabla_{\bar{\partial}_a} \delta_b = C^c{}_{ab} \delta_c , \; \nabla_{\bar{\partial}_a} \bar{\partial}_b = C^c{}_{ab} \bar{\partial}_c ,$$

$$F^c{}_{ab} = \frac{1}{2} g^{F\,cd}(\delta_a g^F_{bd} + \delta_b g^F_{ad} - \delta_d g^F_{ab}) ,$$

$$C^c{}_{ab} = \frac{1}{2} g^{F\,cd}(\bar{\partial}_a g^F_{bd} + \bar{\partial}_b g^F_{ad} - \bar{\partial}_d g^F_{ab}) .$$

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Introduction to Cartan geometry

- Cartan geometry modeled on Klein geometry G/H:
  - G and H are Lie groups.
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- Principal H-bundle $\pi : P \rightarrow M$:
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- Cartan connection $A \in \Omega^1(P, \mathfrak{g})$ satisfying:
  1. For each $p \in P$, $A_p = A|_{T_pP} : T_pP \to \mathfrak{g}$ is a linear isomorphism.
  2. A is H-equivariant: $(R_h)^*A = \mathrm{Ad}(h^{-1}) \circ A$ for all $h \in H$.
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  - $\Rightarrow$ Dimension of the base manifold: $\dim M = \dim G - \dim H = \dim G/H$.
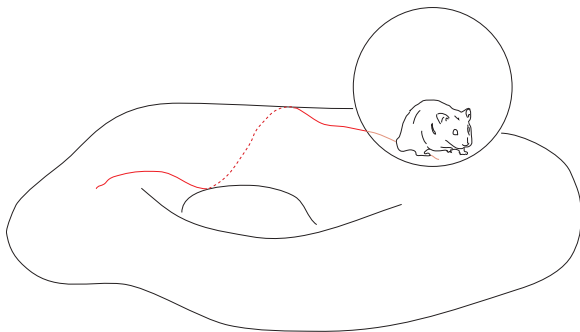
Toy model for Cartan geometry: The hamster ball
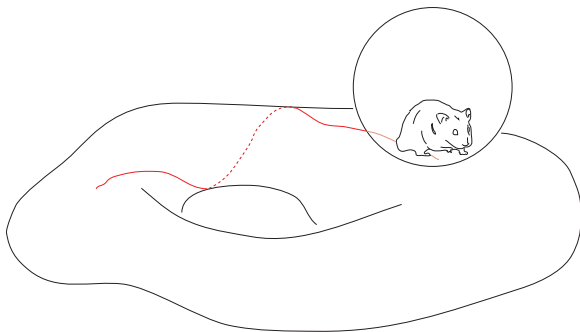
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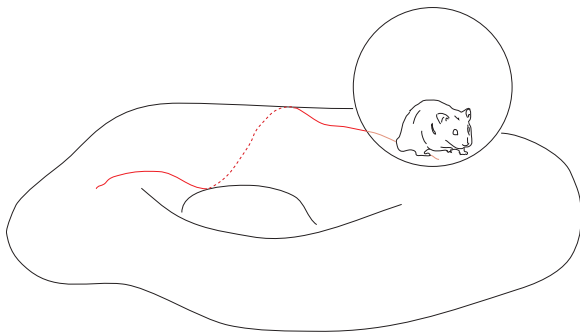
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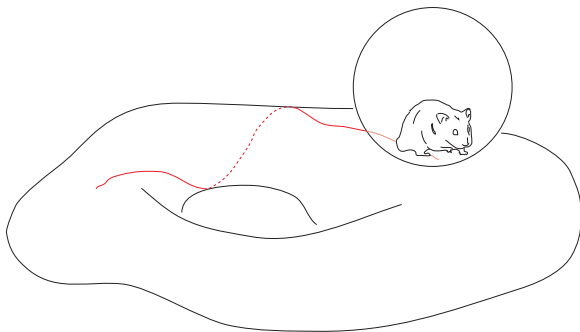
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  - Rotations around its position $x = \pi(p)$: subalgebra $\mathfrak{h} = \mathfrak{so}(2)$.
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- Surface M "traced" by $S^2 \sim SO(3)/SO(2) = G/H$

Klein geometries for spacetime and observer space

- Consider groups $G \supset H \supset K$:
  - "Inhomogeneous group" - symmetry group of homogeneous space:

  $$G_\Lambda = \begin{cases} \mathrm{SO}_0(4,1) & \Lambda = 1 \\ \mathrm{ISO}_0(3,1) & \Lambda = 0 \\ \mathrm{SO}_0(3,2) & \Lambda = -1 \end{cases}.$$

  - "Homogeneous group" $H = \mathrm{SO}_0(3,1)$ - stabilizer of a point.
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- Induced split of Lie algebra $\mathfrak{g}$ via Ad:
  - Irreducible representations of $H \subset G$ on $\mathfrak{g}$:

$$\mathfrak{g} = \underbrace{\mathfrak{h}}_{\text{Lorentz transformations}} \oplus \underbrace{\mathfrak{z}}_{\text{translations}}.$$

  - Irreducible representations of $K \subset G$ on $\mathfrak{g}$:

$$\mathfrak{h} = \underbrace{\mathfrak{k}}_{\text{rotations}} \oplus \underbrace{\mathfrak{n}}_{\text{boosts}}, \quad \mathfrak{z} = \underbrace{\vec{\mathfrak{z}}}_{\text{spatial translations}} \oplus \underbrace{\mathfrak{z}^0}_{\text{temporal translations}}.$$

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- $\Rightarrow$ Geometry of M encoded in A resp. $\underline{A}$.

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$$\mathfrak{g} = \mathfrak{k} + \mathfrak{y} + \vec{\mathfrak{z}} + \mathfrak{z}^0$$

- Infinitesimal rotations $\in R_p P \cong \mathfrak{k}$.
- Infinitesimal Lorentz boosts $\in B_p P \cong \mathfrak{y}$.
- Infinitesimal spatial translations $\in \vec{H}_p P \cong \vec{\mathfrak{z}}$.
- Infinitesimal temporal translations $\in H_p^0 P \cong \mathfrak{z}^0$.

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\begin{array}{ccccccccc}
T_p P & = & R_p P & + & B_p P & + & \vec{H}_p P & + & H_p^0 P \\
A_p \uparrow & = & \Omega_p \uparrow & + & b_p \uparrow & + & \vec{e}_p \uparrow & + & e_p^0 \uparrow \\
\mathfrak{g} & = & \mathfrak{k} & + & \mathfrak{y} & + & \vec{\mathfrak{z}} & + & \mathfrak{z}^0
\end{array}
$$

   ○ Infinitesimal rotations $\in R_p P \cong \mathfrak{k}$.
   ○ Infinitesimal Lorentz boosts $\in B_p P \cong \mathfrak{y}$.
   ○ Infinitesimal spatial translations $\in \vec{H}_p P \cong \vec{\mathfrak{z}}$.
   ○ Infinitesimal temporal translations $\in H_p^0 P \cong \mathfrak{z}^0$.

- *Cartan connection* $A = \Omega + b + \vec{e} + e^0 \in \Omega^1(P, \mathfrak{g})$.
- *Fundamental vector fields* $\underline{A} : \mathfrak{g} \to \Gamma(TP)$ as "inverse" of A.

⇒ Geometry of M encoded in A resp. $\underline{A}$. [S. Gielen, D. Wise '12]

Outline

1. Introduction

2. Geometries

3. Application in physics

4. Conclusion

From pseudo-Riemannian to Finsler

- Metric-induced 2-homogeneous geometry function:

$$L(x, y) = g_{ab}(x)y^a y^b.$$

$\Rightarrow$ Finsler function $F(x, y) = \sqrt{|L(x, y)|}$.

$\Rightarrow$ Finsler metric

$$g^F(x, y) = \begin{cases} -g(x, y) & \text{for } y \text{ timelike,} \\ g(x, y) & \text{for } y \text{ spacelike.} \end{cases}$$

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$\Rightarrow$ Cartan non-linear connection:

$$N^a{}_b = \Gamma^a{}_{bc} y^c.$$

$\Rightarrow$ Cartan linear connection:

$$F^a{}_{bc} = \Gamma^a{}_{bc}, \quad C^a{}_{bc} = 0.$$

From Finsler to Cartan

- Need to construct $A \in \Omega^1(P, \mathfrak{g})$.
- Recall that

$$
\begin{array}{rcccc}
\mathfrak{g} & = & \mathfrak{h} & \oplus & \mathfrak{z} \\
A & = & \omega & + & e
\end{array}
$$

- Definition of e: Use the *solder form*:

$$
e^i = f^{-1i}_{a} dx^a .
$$

- Definition of ω: Use the *Cartan linear connection*:

$$
\omega^i_{j} = f^{-1i}_{a} \left[df^a_j + f^b_j \left(dx^c F^a_{bc} + (dx^d N^c_{d} + df^c_0) C^a_{bc} \right) \right] .
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- Let $a = z^i \mathcal{Z}_i + \frac{1}{2} h^i_{\;j} \mathcal{H}_i^{\;j} \in \mathfrak{g}$.
- Fundamental vector fields:

$$\underline{A}(a) = z^i f^a_i \left(\partial_a - f^b_j F^c_{\;ab} \bar{\partial}^j_c \right) + \left(h^i_{\;j} f^a_i - h^i_{\;0} f^b_i f^c_j C^a_{\;bc} \right) \bar{\partial}^j_a.$$

Relation between geometries

Metric geometry

Manifold M

Lorentzian metric g

Orientation

Time orientation

Finsler geometry

Tangent bundle TM

Geometry function
$L : TM \to \mathbb{R}$

Finsler function $F : TM \to \mathbb{R}$

Finsler metric $g^F(x, y)$

Cartan non-linear connection
$N^a{}_b$

Cartan linear connection ∇

Cartan geometry

Lie group $G = \mathrm{ISO}_0(3, 1)$

Closed subgroup
$K = \mathrm{SO}(3)$

Principal K-bundle
$\pi : P \to O$

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From metric to Finsler

Coordinates (x^a) on M

Coordinates (x^a, y^a) on TM

Define $L(x, y) = g_{ab}(x) y^a y^b$

From Finsler to Cartan

Space O of observer 4-velocities

Space P of observer frames

Define A from connection ∇

Outline
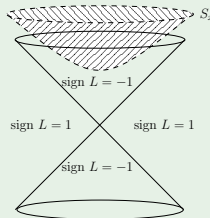
Outline

Causal structure

Metric geometry

Geometry function:

$$L = g_{ab}y^a y^b$$
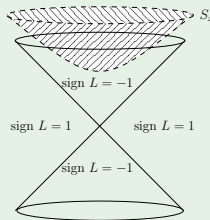
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Finsler geometry

Fundamental geometry function L

Hessian:

$$g_{ab}^L(x,y) = \frac{1}{2}\bar{\partial}_a\bar{\partial}_b L(x,y)$$

Use sign of L and signature of g^L.

Causal structure

Metric geometry

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S_x

sign $L = -1$

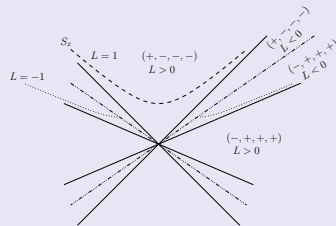sign $L = 1$ sign $L = 1$

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Finsler geometry

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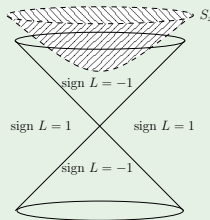Use sign of L and signature of g^L.

S_x $L = 1$ $(+, -, -, -)$ $(+, -, -, -)$
$L > 0$ $L < 0$

$L = -1$ $(-, +, +, +)$
$L < 0$

$(-, +, +, +)$
$L > 0$

Cartan geometry

Observer space:

$$O = \bigcup_{x \in M} S_x$$
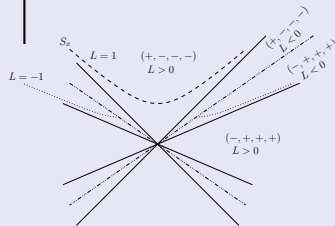
O contains only future unit timelike vectors.

Causality of Finsler spacetimes

- "Unit timelike condition" required for Finsler spacetimes:
  For all $x \in M$ the set

  $$\Omega_x = \left\{ y \in T_x M \,\middle|\, |L(x,y)| = 1, \operatorname{sig} \bar{\partial}_a \bar{\partial}_b L(x,y) = (\epsilon, -\epsilon, -\epsilon, -\epsilon) \right\}$$

  with $\epsilon = L(x,y)/|L(x,y)|$ contains a non-empty closed connected component $S_x \subseteq \Omega_x \subset T_x M$.

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- $\Rightarrow$ S_x contains physical observers.
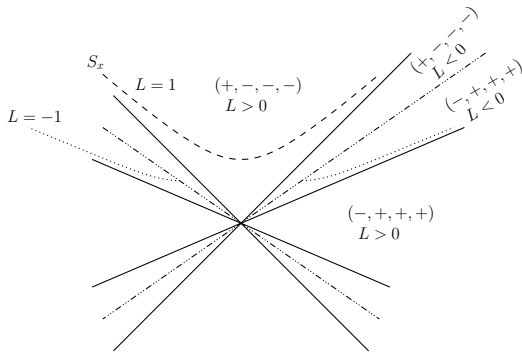- $\Rightarrow$ $\mathbb{R}^+ S_x$ is convex cone.

The observer frame bundle

- Observer space of a Finsler spacetime:
  - Consider all allowed observer tangent vectors:

  $$O = \bigcup_{x \in M} S_x \, .$$

  - Tangent vectors $y \in S_x$ satisfy $g^F_{ab}(x,y)y^a y^b = 1$.

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- Construct orthonormal observer frames:
  - $\Rightarrow$ Complete $y = f_0$ to a frame f_i with $g^F_{ab}(x, y)f^a_i f^b_j = -\eta_{ij}$.
  - Let P be the space of all observer frames.
  - Natural projection $\pi : P \to O$ discards spatial frame components.

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  - Let P be the space of all observer frames.
  - Natural projection $\pi : P \rightarrow O$ discards spatial frame components.
- Group action on the frame bundle:
  - SO(3) acts on spatial frame components by rotations.
  - Action is free and transitive on fibers of $\pi : P \rightarrow O$.
  - $\Rightarrow$ $\pi : P \rightarrow O$ is principal K-bundle.

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Observers

Metric geometry

Timelike curve γ:

$$\gamma : \begin{array}{ccc} \mathbb{R} & \to & M \\ \tau & \mapsto & \gamma(\tau) \end{array}$$

$$g_{ab}\dot{\gamma}^a\dot{\gamma}^b = -1$$

Orthonormal frame f:

$$f_0^a = \dot{\gamma}^a$$

$$g_{ab}f_i^a f_j^b = \eta_{ij}$$

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$$\dot{\gamma}(\tau) \in S_{\gamma(\tau)} \subset TM$$

Canonical lift Γ:

$$\Gamma(\tau) = (\gamma(\tau), \dot{\gamma}(\tau))$$

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Cartan geometry

Observer curve Γ:

$$\Gamma \;:\; \mathbb{R} \;\to\; O$$
$$\tau \;\mapsto\; \Gamma(\tau)$$

Lift condition:

$$\tilde{e}^i\dot{\Gamma}(\tau) = \delta_0^i$$

Orthonormal frame f:

$$f \in \pi^{-1}(\Gamma(\tau)) \subset P$$

Inertial observers

Metric geometry

Minimize arc length integral:

$$\int_{t_1}^{t_2} \sqrt{|g_{ab}(\gamma(t))\dot{\gamma}^a(t)\dot{\gamma}^b(t)|}\, dt$$

Geodesic equation:

$$\ddot{\gamma}^a + \Gamma^a{}_{bc}\dot{\gamma}^b\dot{\gamma}^c = 0$$

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Geodesic spray:

$$\mathbf{S} = y^a(\partial_a - N^b{}_a\bar\partial_b)$$

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$$\dot\Gamma(\tau) = \mathbf{S}(\Gamma(\tau))$$

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Observers on Finsler spacetimes

- Observer trajectories and canonical lifts:
  - Observer trajectory γ in *M*.
  - Lift γ to a curve $\Gamma = (\gamma, \dot\gamma)$ in *TM*.
  - Curves Γ in *TM* are canonical lifts if and only if

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  $\Rightarrow$ Γ is integral curve of geodesic spray:

  $$\dot{\Gamma} = \mathbf{S} = y^a \delta_a.$$

Observers on Cartan observer space

- Observer curves:
  - Consider curve Γ in *O*.
  - ⇒ Tangent vector splits into translation and boost:

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- Translational component of the tangent vector:
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  - Components are relative to observer's frame.
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- Boost component of the tangent vector:
  - Measures acceleration in observer's frame.
  - Inertial observers are non-accelerating: $b^\alpha \dot{\Gamma} = 0$.
  - $\Rightarrow$ Inertial observers follow integral curves of time translation: $\dot{\Gamma} = e_0$.

Observers from Finsler to Cartan

- Generating vector field on Finsler spacetimes:
  - Geodesic spray **S** preserves Finsler function: $\mathbf{S}F = 0$.
  - $\Rightarrow$ Geodesic spray **S** is tangent to observer space O (level set).
  - $\rightsquigarrow$ Define Reeb vector field $\mathbf{r} = \mathbf{S}|_O$.
  - Coordinate expression: $\mathbf{r} = y^a(\partial_a - N^b{}_a \bar{\partial}_b)$.

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- $\Rightarrow$ Cartan trajectories correspond to Finslerian parallel transport.

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Einstein-Hilbert action:

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Einstein-Hilbert action:
$$S_{\text{EH}} = \frac{1}{2\kappa} \int_M d^4x \sqrt{-g}\, R$$

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Using non-linear connection:

$$S_N = \frac{1}{\kappa} \int_\Sigma \text{Vol}_{\tilde{G}}\, R^a{}_{ab} y^b$$

Using linear connection:

$$S_L = \frac{1}{\kappa} \int_\Sigma \text{Vol}_{\tilde{G}}\, g^{F\,ab} R^c{}_{acb}$$

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Einstein-Hilbert action:
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Cartan geometry

Using horizontal vector fields:
$$S_{\text{H}} = \int_O \tilde{b}^\alpha([\underline{\tilde{e}}_\alpha, \underline{\tilde{e}}_0])\, \text{Vol}_O$$

Using Cartan curvature:
$$S_{\text{C}} = \int_O \kappa_{\mathfrak{h}}(\tilde{F}_{\mathfrak{h}} \wedge \tilde{F}_{\mathfrak{h}}) \wedge \text{Vol}_S$$

Gravity

Metric geometry

Einstein-Hilbert action:
$$S_{\text{EH}} = \frac{1}{2\kappa} \int_M d^4x \sqrt{-g}\, R$$

Finsler geometry

Using non-linear connection:
$$S_N = \frac{1}{\kappa} \int_\Sigma \text{Vol}_{\tilde{G}}\, R^a{}_{ab} y^b$$

Using linear connection:
$$S_L = \frac{1}{\kappa} \int_\Sigma \text{Vol}_{\tilde{G}}\, g^{F\,ab} R^c{}_{acb}$$

$\Leftrightarrow$

$\Leftrightarrow$

Cartan geometry

Using horizontal vector fields:
$$S_H = \int_O \tilde{b}^\alpha([\underline{\tilde{e}}_\alpha, \underline{\tilde{e}}_0]) \,\text{Vol}_O$$

Using Cartan curvature:
$$S_C = \int_O \kappa_{\mathfrak{h}}(\tilde{F}_{\mathfrak{h}} \wedge \tilde{F}_{\mathfrak{h}}) \wedge \text{Vol}_S$$

Gravity from Cartan to Finsler

- MacDowell-Mansouri gravity on observer space: [S. Gielen, D. Wise '12]

$$S_G = \int_O \epsilon_{\alpha\beta\gamma} \, \mathrm{tr}_{\mathfrak{h}}(F_{\mathfrak{h}} \wedge \star F_{\mathfrak{h}}) \wedge b^\alpha \wedge b^\beta \wedge b^\gamma$$

- ○ Hodge operator $\star$ on $\mathfrak{h}$.
- ○ Non-degenerate H-invariant inner product $\mathrm{tr}_{\mathfrak{h}}$ on $\mathfrak{h}$.
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  - Boost part $b \in \Omega_1(P, \mathfrak{y})$ of the Cartan connection.
- Translate terms into Finsler language (with $R = d\omega + \frac{1}{2}[\omega, \omega]$):
  - Curvature scalar:

$$[e, e] \wedge \star R \rightsquigarrow g^{F\,ab} R^c{}_{acb} \, dV \,.$$

  - Cosmological constant:

$$[e, e] \wedge \star [e, e] \rightsquigarrow dV \,.$$

  - Gauss-Bonnet term:

$$R \wedge \star R \rightsquigarrow \epsilon^{abcd} \epsilon^{efgh} R_{abef} R_{cdgh} \, dV \,.$$

⇒ Gravity theory on Finsler spacetime.

Gravity from Finsler to Cartan

- Finsler gravity action: [C. Pfeifer, M. Wohlfarth '11]

$$S_G = \int_O d^4x \, d^3y \, \sqrt{-\tilde{G}} R^a{}_{ab} y^b.$$

  - Sasaki metric $\tilde{G}$ on O.
  - Non-linear curvature $R^a{}_{ab}$.

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  ○ Non-linear curvature $R^a{}_{ab}$.

- Translate terms into Cartan language:

$$d^4x \, d^3y \, \sqrt{-\tilde{G}} = \epsilon_{ijkl}\epsilon_{\alpha\beta\gamma} \, e^i \wedge e^j \wedge e^k \wedge e^l \wedge b^\alpha \wedge b^\beta \wedge b^\gamma \,,$$
$$R^a{}_{ab} y^b = b^\alpha [\underline{A}(\mathcal{Z}_\alpha), \underline{A}(\mathcal{Z}_0)] \,.$$

⇒ Gravity theory on observer space.

Outline

Summary

- Finsler spacetimes
  - Generalization of pseudo-Riemannian spacetimes.
  - Geometry defined by function L on TM.
  - Lengths measured by Finsler function $F = |L|^{\frac{1}{n}}$.
  - Metric generalized by Finsler metric g_{ab}^F.

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  - Can be obtained from Finsler spacetimes.
  - Geometry on principal SO(3)-bundle $\pi : P \to O$.
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- Different geometries provide compatible definitions of:
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  - Observers
  - Observables
  - Gravity

Caveats and outlook

- Observer space not most suitable for Lagrange theory:
  - Lagrangian defined on jet bundle over configuration bundle.
  - Critical sections: solutions of Euler-Lagrange equations.
  - Euler-Lagrange equations determined from variational calculus.
  - Variational calculus assumes fixed configuration bundle.
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- Cartan geometry version of projective bundle approach?

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