The gauge-invariant parametrized post-Newtonian formalism

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Motivation

- Experimental tests of modified gravity theories:
  - Cosmological observations (CMB, supernovae, ...).
  - Gravitational waves.
  - Direct observation of black holes.
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  - More fundamental fields constituting the metric.
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$\rightsquigarrow$ Use gauge-invariant higher order perturbation theory.

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⇝ Introduce a *gauge*: diffeomorphism $\mathcal{X} : M_0 \to M$.

1. Identification of (coordinated) points on M and M_0.
2. Comparison between reference metric $g^{(0)}$ and ${}^{\mathcal{X}}g = \mathcal{X}^*g$ on M_0.

Gauge and perturbations

- Parameter dependent physical metric:
  - Assume physical metric $g \equiv g_\epsilon$ depends on parameter $\epsilon \in \mathbb{R}$.
  - Assume every g_ϵ is defined on its own M_ϵ.
  - Assume $g_0 = g^{(0)}$ is the reference metric on M_0.
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{}^{\mathcal{X}}g_\epsilon = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \left.\frac{\partial^k\, {}^{\mathcal{X}}g_\epsilon}{\partial \epsilon^k}\right|_{\epsilon=0} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!}\, {}^{\mathcal{X}}g^{(k)} \,.
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  - Series coefficients ${}^{\mathcal{X}}g^{(k)}$ depend on gauge choice $\mathcal{X}$.

Gauge invariant perturbations

- Choose a fixed "distinguished" gauge $\mathcal{S}_\epsilon : M_0 \to M_\epsilon$:
  - E.g., impose gauge conditions on the metric.
  - Examples: harmonic gauge used for GWs, standard PPN gauge.

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- Metric in arbitrary gauge $\mathcal{X}$:

$$ {}^{\mathcal{X}}g_\epsilon = \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} \cdots \frac{\epsilon^{l_1+\cdots+kl_k+\cdots}}{(1!)^{l_1}\cdots(k!)^{l_k}\cdots l_1!\cdots l_k!\cdots} \pounds_{X_{(1)}}^{l_1} \cdots \pounds_{X_{(k)}}^{l_k} \cdots \mathbf{g}_\epsilon \,. $$

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  - Gauge defining vector fields $X_{(k)}$: coordinate choice.
- Number # of independent components:

$$
\#({}^{\mathcal{X}}g_\epsilon) = \#(\mathbf{g}_\epsilon) + \#(X_{(k)})\,.
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Outline

Post-Newtonian matter and velocity orders

- Energy-momentum tensor of a perfect fluid:

$$T^{\mu\nu} = (\rho + \rho\Pi + p)u^{\mu}u^{\nu} + pg^{\mu\nu} \,.$$

  - Rest mass density ρ.
  - Specific internal energy Π.
  - Pressure p.
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- Universe rest frame and slow-moving source matter:
  - Consider some gauge $\mathcal{X} : M_0 \to M$ ("universe rest frame").
  - Pullback of metric and matter variables along $\mathcal{X}$.
  - Velocity of the source matter: ${}^{\mathcal{X}}v^i = {}^{\mathcal{X}}u^i / {}^{\mathcal{X}}u^0$.
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- Quasi-static: assign additional $\mathcal{O}(1)$ to time derivatives ∂_0.

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{}^{\mathcal{P}}\overset{2}{g}_{ij} &= 2\gamma{}^{\mathcal{P}}U\delta_{ij}\,, \\
{}^{\mathcal{P}}\overset{3}{g}_{0i} &= -\frac{1}{2}(3+4\gamma+\alpha_1-\alpha_2+\zeta_1-2\xi){}^{\mathcal{P}}V_i-\frac{1}{2}(1+\alpha_2-\zeta_1+2\xi){}^{\mathcal{P}}W_i\,, \\
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  - PPN potentials describe source matter distribution.
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$\rightsquigarrow$ Decompose metric into gauge-invariant and pure gauge parts.

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Gauge-invariant metric

- Definition of gauge-invariant metric components:

$$\mathbf{g}_{00} = \mathbf{g}^{\star}\,, \quad \mathbf{g}_{0i} = \mathbf{g}^{\diamond}_{i}\,, \quad \mathbf{g}_{ij} = \mathbf{g}^{\bullet}\delta_{ij} + \mathbf{g}^{\dagger}_{ij}\,.$$

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- Conditions imposed on components:
$$\partial^{i}\mathbf{g}^{\diamond}_{i} = 0\,, \quad \partial^{i}\mathbf{g}^{\dagger}_{ij} = 0\,, \quad \mathbf{g}^{\dagger}_{[ij]} = 0\,, \quad \mathbf{g}^{\dagger}_{ii} = 0\,.$$

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- Relation to arbitrary gauge $\mathcal{X}$:

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- Gauge defining vector fields:

$$X_i = \partial_i X^{\blacklozenge} + X^{\diamond}_{i}\,, \quad X_0 = X^{\star}\,, \quad \partial^i X^{\diamond}_{i} = 0\,.$$

Decomposition of metric components

- Count number of independent components at each order:

total		invariant		pure gauge	
$\mathcal{X}\,\overset{2}{g}_{00}$	1	$\overset{2}{\mathbf{g}}{}^{\star}$	1	-	0
$\mathcal{X}\,\overset{2}{g}_{ij}$	6	$\overset{2}{\mathbf{g}}{}^{\star}, \overset{2}{\mathbf{g}}{}^{\dagger}_{ij}$	$1+2$	$\overset{2}{X}{}^{\blacklozenge}, \overset{2}{X}{}^{\diamond}_{i}$	$1+2$
$\mathcal{X}\,\overset{3}{g}_{0i}$	3	$\overset{3}{\mathbf{g}}{}^{\diamond}_{i}$	2	$\overset{3}{X}{}^{\star}$	1
$\mathcal{X}\,\overset{4}{g}_{00}$	1	$\overset{4}{\mathbf{g}}{}^{\star}$	1	-	0
$\mathcal{X}\,\overset{4}{g}_{ij}$	6	$\overset{4}{\mathbf{g}}{}^{\star}, \overset{4}{\mathbf{g}}{}^{\dagger}_{ij}$	$1+2$	$\overset{4}{X}{}^{\blacklozenge}, \overset{4}{X}{}^{\diamond}_{i}$	$1+2$

Decomposition of metric components

- Count number of independent components at each order:

total		invariant		pure gauge	
${}^{\mathcal{X}}\overset{2}{g}_{00}$	1	$\overset{2}{\mathbf{g}}^{\star}$	1	-	0
${}^{\mathcal{X}}\overset{2}{g}_{ij}$	6	$\overset{2}{\mathbf{g}}^{\star}, \overset{2}{\mathbf{g}}^{\dagger}_{ij}$	$1 + 2$	$\overset{2}{X}^{\blacklozenge}, \overset{2}{X}^{\diamond}_{i}$	$1 + 2$
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$\Rightarrow$ Components split into invariant and gauge parts.

$\Rightarrow$ Possible to separate physical information from coordinate choice.

Relation to standard PPN gauge

- Use relation between expansion coefficients:

$$\overset{\mathcal{P}}{\underset{k}{g}} = \sum_{0 \leq l_1 + 2l_2 + \ldots \leq k} \frac{1}{l_1!\, l_2! \cdots} \underset{\substack{1\\P}}{\pounds}^{l_1} \cdots \underset{\substack{k\\P}}{\pounds}^{l_k} \cdots \overset{k - l_1 - 2l_2 - \ldots}{\mathbf{g}}.$$

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⇒ Gauge defining vector fields:

$$ \overset{2}{P}{}^{\blacklozenge} = 0 , \quad \overset{2}{P}{}^{\diamond}_{i} = 0 , \quad \overset{3}{P}{}^{\star} = -\frac{1}{4}(2 + 4\gamma + \alpha_1 - 2\alpha_2 + 2\zeta_1 - 4\xi)\chi_{,0} . $$

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$$\begin{aligned}\overset{4}{\mathbf{g}}{}^{\star} = {} & \frac{1}{2}(2-\alpha_1+2\alpha_2+2\alpha_3)\mathbf{\Phi}_1 + 2(1+3\gamma-2\beta+\zeta_2+\xi)\mathbf{\Phi}_2 \\ & + 2(1+\zeta_3)\mathbf{\Phi}_3 + 2(3\gamma+3\zeta_4-2\xi)\mathbf{\Phi}_4 - 2\xi\mathbf{\Phi}_W - 2\beta\mathbf{U}^2 \\ & + \frac{1}{2}(2+4\gamma+\alpha_1-2\alpha_2)\mathfrak{A} + \frac{1}{2}(2+4\gamma+\alpha_1-2\alpha_2+2\zeta_1-4\xi)\mathfrak{B}\,.\end{aligned}$$

Gauge-invariant field equations

- Perform similar decomposition of energy-momentum tensor:

$$\mathbf{T}^{\star} = \mathbf{T}_{00} = \boldsymbol{\rho}\left(1 - \overset{2}{\mathbf{g}}_{00} + \mathbf{v}^2 + \boldsymbol{\Pi}\right) + \mathcal{O}(6)\,,$$

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$$\overset{4}{\mathbf{T}}{}^{\bullet} = \frac{1}{3}\boldsymbol{\rho}\mathbf{v}^2 + \mathbf{p} = -\frac{1}{12\pi}\triangle(\boldsymbol{\Phi}_1 + 3\boldsymbol{\Phi}_4)\,, \quad \overset{4}{\mathbf{T}}{}^{\blacktriangle} = \frac{1}{16\pi}(3\mathfrak{A} - \boldsymbol{\Phi}_1)\,.$$

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$\Rightarrow$ Find PPN parameters by comparing coefficients on both sides.

Outline

Action and field equations

- Action of scalar-tensor gravity with massless scalar field: [Nordtvedt '70]

$$S = \frac{1}{2\kappa^2}\int_M d^4x\sqrt{-g}\left(\psi R - \frac{\omega(\psi)}{\psi}\partial_\rho\psi\partial^\rho\psi\right) + S_m[g_{\mu\nu},\chi]\,.$$

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$\rightsquigarrow$ Decompose into gauge-invariant field equations.

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$\Rightarrow$ Remaining equations determine gauge-invariant metric.

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- Normalization of the gravitational constant:

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⇒ PPN parameters reproduce well-known result: [Nordtvedt '70]

$$\gamma = \frac{\omega_0 + 1}{\omega_0 + 2}\,, \quad \beta = 1 + \frac{\omega_1\Psi}{4(2\omega_0 + 3)(\omega_0 + 2)^2}\,,$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \xi = 0\,.$$

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Summary

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  - Distinguish between physical and background spacetime.
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- Post-Newtonian limit of scalar-tensor gravity:
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  - Consistency check: obtain well-known PPN parameters.
  - Also possible to use tetrad formulation to calculate solution.

Outlook

- Extend formalism by including higher perturbation orders:
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    - Take into account Vainshtein or other screening mechanisms.
    - Cosmological background and time variability of PPN parameters.
    - Include PPN potentials appearing in higher derivative theories.
- Apply formalism to complicated gravity theories:
    - Bimetric and multimetric gravity theories.
    - Multi-scalar Horndeski generalizations.
    - Theories involving generalized Proca fields.
    - Extensions based on metric-affine geometry.
    - Extensions of teleparallel and symmetric teleparallel gravity.

Literature

Further reading

MH,
"Gauge invariant approach to the parametrized post-Newtonian formalism",
Phys. Rev. D 101 (2020) 024061 [arXiv:1910.09245].