(MULTI-)SCALAR-TORSION THEORIES OF GRAVITY

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Definitions

The *fundamental fields* are

- a *coframe field*

$$\theta^a = \theta^a{}_\mu \mathrm{d}x^\mu\,, \qquad (1)$$

- a flat *spin connection*

$$\overset{\bullet}{\omega}{}^a{}_b = \overset{\bullet}{\omega}{}^a{}_{b\mu} \mathrm{d}x^\mu\,, \qquad (2)$$

- N *scalar fields* ϕ^A,
- arbitrary *matter fields* χ^I.

These fields further define

- a *frame field* $e_a = e_a{}^\mu \partial_\mu$ with

$$\iota_{e_a}\theta^b = \delta^b_a\,, \qquad (3)$$

a *metric*

$$g_{\mu\nu} = \eta_{ab}\theta^a{}_\mu \theta^b{}_\nu\,, \qquad (4)$$

- a *volume form*

$$\theta \mathrm{d}^4x = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3\,, \qquad (5)$$

- the *Levi-Civita* connection

$$\overset{\circ}{\omega}_{ab} = -\frac{1}{2}(\iota_{e_b}\iota_{e_c}\mathrm{d}\theta_a + \iota_{e_c}\iota_{e_a}\mathrm{d}\theta_b - \iota_{e_a}\iota_{e_b}\mathrm{d}\theta_c)\theta^c\,, \qquad (6)$$

- the *torsion*

$$T^a = \mathrm{d}\theta^a + \overset{\bullet}{\omega}{}^a{}_b \wedge \theta^b\,. \qquad (7)$$

References

[1] M. Hohmann, “Scalar-torsion theories of gravity I: general formalism and conformal transformations,” arXiv:1801. [gr-qc].

[2] M. Hohmann, C. Pfeifer, “Scalar-torsion theories of gravity II: $L(T, X, Y, \phi)$ theory,” arXiv:1801. [gr-qc].

[3] M. Hohmann, “Scalar-torsion theories of gravity III: analogue of scalar-tensor gravity and conformal invariants,” arXiv:1801. [gr-qc].

[4] M. Hohmann, L. Järv, U. Ualikhanova, “Covariant formulation of scalar-torsion gravity,” arXiv:1801.05786 [gr-qc].

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General theory [1]

We consider an *action* of the generic form

$$S\left[\theta^a, \overset{\bullet}{\omega}{}^a{}_b, \phi^A, \chi^I\right] = S_g\left[\theta^a, \overset{\bullet}{\omega}{}^a{}_b, \phi^A\right] + S_m\left[\theta^a, \phi^A, \chi^I\right]\,, \qquad (8)$$

Its *variation* can be written, after integration by parts, in the form

$$\delta S_g = \int_M \left(\Delta_a \wedge \delta\theta^a + \frac{1}{2}\Xi_a{}^b \wedge \delta\overset{\bullet}{\omega}{}^a{}_b + \Phi_A \wedge \delta\phi^A\right) = \int_M \left(\Upsilon_a \wedge \delta\theta^a + \Pi_a \wedge \delta T^a + \Phi_A \wedge \delta\phi^A\right)\,, \qquad (9a)$$

$$\delta S_m = \int_M \left(\Sigma_a \wedge \delta\theta^a + \Psi \wedge \delta\phi + \Omega_I \wedge \delta\chi^I\right)\,, \qquad (9b)$$

where the terms in the two alternative forms of δS_g are related by

$$\Delta_a = \Upsilon_a - \overset{\bullet}{\mathrm{D}}\Pi_a \quad \Leftrightarrow \quad \Upsilon^a = \Delta^a + \overset{\bullet}{\mathrm{D}}\left(\frac{1}{4}\iota_{e_c}\iota_{e_b}\Xi^{bc} \wedge \theta^a - \iota_{e_b}\Xi^{ab}\right)\,, \qquad (10a)$$

$$\Xi^{ab} = -2\Pi^{[a} \wedge \theta^{b]} \quad \Leftrightarrow \quad \Pi^a = \frac{1}{4}\iota_{e_c}\iota_{e_b}\Xi^{bc} \wedge \theta^a - \iota_{e_b}\Xi^{ab}\,. \qquad (10b)$$

A particular variation of the action is generated by *local Lorentz transformations* $\lambda^a{}_b$, under which the fields change according to

$$\delta_\lambda\theta^a = \lambda^a{}_b\theta^b\,, \quad \delta_\lambda\overset{\bullet}{\omega}{}^a{}_b = \lambda^a{}_c\overset{\bullet}{\omega}{}^c{}_b - \overset{\bullet}{\omega}{}^a{}_c\lambda^c{}_b - \mathrm{d}\lambda^a{}_b = -\overset{\bullet}{\mathrm{D}}\lambda^a{}_b \quad \Rightarrow \quad \delta_\lambda T^a = \lambda^a{}_b T^b\,. \qquad (11)$$

Demanding local Lorentz invariance, $\delta_\lambda S_g = \delta_\lambda S_m = 0$, then yields $\Sigma^{[a} \wedge \theta^{b]} = 0$ and

$$\Upsilon^{[a} \wedge \theta^{b]} + \Pi^{[a} \wedge T^{b]} = 0 \quad \Leftrightarrow \quad \Delta^{[a} \wedge \theta^{b]} - \frac{1}{2}\overset{\bullet}{\mathrm{D}}\Xi^{ab} = 0\,. \qquad (12)$$

From diffeomorphism invariance of the matter action follows *energy-momentum conservation*

$$\overset{\circ}{\mathrm{D}}\Sigma_a + \Psi_A \wedge \iota_{e_a}\mathrm{d}\phi^A = 0\,. \qquad (13)$$

The *field equations* are $\Omega_I = 0$ for matter, $\Phi_A + \Psi_A = 0$ for the scalar fields and

$$\Delta_a + \Sigma_a = 0 \quad \Leftrightarrow \quad \Upsilon_a - \overset{\bullet}{\mathrm{D}}\Pi_a + \Sigma_a = 0 \qquad (14)$$

for the tetrads; the antisymmetric part of the latter agrees with the connection equations

$$\overset{\bullet}{\mathrm{D}}\Xi^{ab} = 0 \quad \Leftrightarrow \quad \overset{\bullet}{\mathrm{D}}\Pi^{[a} \wedge \theta^{b]} + \Pi^{[a} \wedge T^{b]} = 0\,. \qquad (15)$$

If we apply a *conformal transformation* of the form

$$\bar{\theta}^a = e^{\gamma(\phi)}\theta^a\,, \quad \bar{e}_a = e^{-\gamma(\phi)}e_a\,, \quad \bar{\phi}^A = f^A(\phi)\,, \qquad (16)$$

then we can find a different action $\bar{S}_g$ and $\bar{S}_m$ of the new field variables, for which holds

$$\bar{S}_g\left[\bar{\theta}^a, \overset{\bullet}{\omega}{}^a{}_b, \bar{\phi}^A\right] = S_g\left[\theta^a, \overset{\bullet}{\omega}{}^a{}_b, \phi^A\right]\,, \quad \bar{S}_m\left[\bar{\theta}^a, \bar{\phi}^A, \chi^I\right] = S_m\left[\theta^a, \phi^A, \chi^I\right]\,, \qquad (17)$$

if the variation of the original and transformed action are related by

$$\bar{\Delta}_a = e^{-\gamma}\Delta_a\,, \quad \bar{\Xi}_a{}^b = \Xi_a{}^b\,, \quad \bar{\Phi}_A = \frac{\partial\phi^B}{\partial\bar{\phi}^A}(\Phi_B - \gamma_{,B}\Delta_a \wedge \theta^a)\,, \qquad (18)$$

which can alternatively be written as

$$\bar{\Upsilon}_a = e^{-\gamma}\left(\Upsilon_a - \gamma_{,A}\Pi_a \wedge \mathrm{d}\phi^A\right)\,, \quad \bar{\Pi}_a = e^{-\gamma}\Pi_a\,, \quad \bar{\Phi}_A = \frac{\partial\phi^B}{\partial\bar{\phi}^A}\left[\Phi_B - \gamma_{,B}\left(\Upsilon_a - \overset{\bullet}{\mathrm{D}}\Pi_a\right) \wedge \theta^a\right]\,, \qquad (19)$$

and finally the matter part

$$\bar{\Sigma}_a = e^{-\gamma}\Sigma_a\,, \quad \bar{\Psi}_A = \frac{\partial\phi^B}{\partial\bar{\phi}^A}(\Psi_B - \gamma_{,B}\Sigma_a \wedge \theta^a)\,, \quad \bar{\Omega}_I = \Omega_I\,. \qquad (20)$$

The new action belongs to the same class of theories.

Theories without derivative couplings [4]

Another subclass of the $L(T, X, Y, \phi)$ theories [2] is given by the *action*

$$S = \frac{1}{2\kappa^2}\int_M \left[f(T, \phi) + Z_{AB}(\phi)g^{\mu\nu}\phi^A_{,\mu}\phi^B_{,\nu}\right]\theta d^4x + S_m[\theta^a, \chi^I]\,. \qquad (21)$$

We call it *minimally coupled* if $f_{T\phi^A} \equiv 0$. The *field equations* are the symmetric part

$$\frac{1}{2}f g_{\mu\nu} + \overset{\circ}{\nabla}_\rho\left(f_T S_{(\mu\nu)}{}^\rho\right) - \frac{1}{2}f_T S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma} - Z_{AB}\phi^A_{,\mu}\phi^B_{,\nu} + \frac{1}{2}Z_{AB}\phi^A_{,\rho}\phi^B_{,\sigma}g^{\rho\sigma}g_{\mu\nu} = \kappa^2\Theta_{\mu\nu}\,, \qquad (22)$$

the antisymmetric part

$$\partial_{[\rho}f_T T^\rho{}_{\mu\nu]} = 0 \quad \Leftrightarrow \quad \partial_\mu f_T\left[\partial_\nu\left(hh_{[a}{}^\mu h_{b]}{}^\nu\right) + 2hh_c{}^{[\mu}h_{[a}{}^{\nu]}\overset{\bullet}{\omega}{}^c{}_{b]\nu}\right] = 0 \qquad (23)$$

and the scalar field equation

$$f_{\phi^A} - \left(2Z_{AB,\phi^C} - Z_{BC,\phi^A}\right)g^{\mu\nu}\phi^B_{,\mu}\phi^C_{,\nu} - 2Z_{AB}\overset{\circ}{\square}\phi^B = 0\,. \qquad (24)$$

There are different ways to *solve the antisymmetric equations* (23):

1. For theories with $f_{TT} \equiv 0$ and $f_{T\phi^A} \equiv 0$, so that $f(T, \phi) = kT - V(\phi)$, the equations (23) are solved identically for any field configuration.
2. Field configurations with $\partial_\mu T = 0$ and $\partial_\mu\phi^A = 0$, i.e., constant torsion scalar and constant scalar fields, always solve the equations (23), independently of the function f. The remaining field equations (22) reduce to general relativity with cosmological constant.
3. Field configurations where T and ϕ^A depend only on a single coordinate y satisfy $\partial_\mu f_T \propto \partial_\mu y$. They solve the equations (23) if the six vector fields, which are defined by the terms in square brackets in (23) for the six values of $[ab]$, are tangent to the hypersurfaces of constant y, independently of the function f.
4. In the general case, the solutions depend on f. An general field configuration solves the equations (23) if the six vector fields mentioned above are tangent to the hypersurfaces of constant f_T.

For the *Friedmann-Lemaître-Robertson-Walker* metric

$$g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right] \qquad (25)$$

one may use the diagonal tetrad

$$h^a{}_\mu = \mathrm{diag}\left(1, \frac{a(t)}{\sqrt{1 - kr^2}}, a(t)r, a(t)r\sin\vartheta\right)\,, \qquad (26)$$

and depending on k one of the spin connections

$$k = 0: \quad \overset{\bullet}{\omega}{}^1{}_{2\vartheta} = -\overset{\bullet}{\omega}{}^2{}_{1\vartheta} = -1\,, \quad \overset{\bullet}{\omega}{}^1{}_{3\varphi} = -\overset{\bullet}{\omega}{}^3{}_{1\varphi} = -\sin\vartheta\,, \quad \overset{\bullet}{\omega}{}^2{}_{3\varphi} = -\overset{\bullet}{\omega}{}^3{}_{2\varphi} = -\cos\vartheta\,; \qquad (27)$$

$$k = 1: \quad \overset{\bullet}{\omega}{}^1{}_{2\vartheta} = -\overset{\bullet}{\omega}{}^2{}_{1\vartheta} = -\sqrt{1 - r^2}\,, \quad \overset{\bullet}{\omega}{}^1{}_{2\varphi} = -\overset{\bullet}{\omega}{}^2{}_{1\varphi} = -r\sin\vartheta\,, \quad \overset{\bullet}{\omega}{}^1{}_{3\vartheta} = -\overset{\bullet}{\omega}{}^3{}_{1\vartheta} = r\,,$$
$$\overset{\bullet}{\omega}{}^1{}_{3\varphi} = -\overset{\bullet}{\omega}{}^3{}_{1\varphi} = -\sqrt{1 - r^2}\sin\vartheta\,, \quad \overset{\bullet}{\omega}{}^2{}_{3r} = -\overset{\bullet}{\omega}{}^3{}_{2r} = -\frac{1}{\sqrt{1 - r^2}}\,, \quad \overset{\bullet}{\omega}{}^2{}_{3\varphi} = -\overset{\bullet}{\omega}{}^3{}_{2\varphi} = -\cos\vartheta\,; \qquad (28)$$

$$k = -1: \quad \overset{\bullet}{\omega}{}^0{}_{1r} = \overset{\bullet}{\omega}{}^1{}_{0r} = \frac{1}{\sqrt{1 + r^2}}\,, \quad \overset{\bullet}{\omega}{}^0{}_{2\vartheta} = \overset{\bullet}{\omega}{}^2{}_{0\vartheta} = r\,, \quad \overset{\bullet}{\omega}{}^0{}_{3\varphi} = \overset{\bullet}{\omega}{}^3{}_{0\varphi} = r\sin\vartheta\,,$$
$$\overset{\bullet}{\omega}{}^1{}_{2\vartheta} = -\overset{\bullet}{\omega}{}^2{}_{1\vartheta} = -\sqrt{1 + r^2}\,, \quad \overset{\bullet}{\omega}{}^1{}_{3\varphi} = -\overset{\bullet}{\omega}{}^3{}_{1\varphi} = -\sqrt{1 + r^2}\sin\vartheta\,, \quad \overset{\bullet}{\omega}{}^2{}_{3\varphi} = -\overset{\bullet}{\omega}{}^3{}_{2\varphi} = -\cos\vartheta \qquad (29)$$

satisfies the third condition listed above.

Theories constructed from four scalar quantities [2]

If we define the *torsion scalar* $T = \frac{1}{2}T^\rho{}_{\mu\nu}S_\rho{}^{\mu\nu}$ through the *superpotential*

$$S_{\rho\mu\nu} = \frac{1}{2}\left(T_{\nu\mu\rho} + T_{\rho\mu\nu} - T_{\mu\nu\rho}\right) - g_{\rho\mu}T^\sigma{}_{\sigma\nu} + g_{\rho\nu}T^\sigma{}_{\sigma\mu}\,, \qquad (30)$$

as well as the scalar field kinetic and coupling terms

$$X^{AB} = -\frac{1}{2}g^{\mu\nu}\phi^A_{,\mu}\phi^B_{,\nu}\,, \quad Y^A = T_\mu{}^{\mu\nu}\phi^A_{,\nu}\,, \qquad (31)$$

then a subclass of scalar-torsion theories [1] is given by the *gravitational action*

$$S_g\left[\theta^a, \overset{\bullet}{\omega}{}^a{}_b, \phi^A\right] = \int_M L\left(T, X^{AB}, Y^A, \phi^A\right)\theta\mathrm{d}^4x\,. \qquad (32)$$

Together with the *matter action variation*

$$\delta S_m[\theta^a, \phi^A, \chi^I] = \int_M \left(\Theta_a{}^\mu\delta\theta^a{}_\mu + \vartheta_A\delta\phi^A + \varpi_I\delta\chi^I\right)\theta\mathrm{d}^4x \qquad (33)$$

we find that the *field equations* are the matter equations $\varpi_I = 0$, the symmetric tetrad equations

$$-Lg_{\mu\nu} - 2\overset{\circ}{\nabla}_\rho\left(L_T S_{(\mu\nu)}{}^\rho\right) + L_T S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma} - L_{X^{AB}}\phi^A_{,\mu}\phi^B_{,\nu}$$
$$+ \overset{\circ}{\nabla}_{(\mu}\left(L_{Y^A}\phi^A_{,\nu)}\right) - \overset{\circ}{\nabla}_\sigma\left(L_{Y^A}\phi^A_{,\rho}\right)g^{\rho\sigma}g_{\mu\nu} + L_{Y^A}\left(T_{(\mu\nu)}{}^\rho\phi^A_{,\rho} + T^\rho{}_{\rho(\mu}\phi^A_{,\nu)}\right) = \Theta_{\mu\nu}\,, \qquad (34)$$

the antisymmetric tetrad / connection equations

$$3\partial_{[\rho}L_T T^\rho{}_{\mu\nu]} + \partial_{[\mu}L_{Y^A}\phi^A_{,\nu]} - \frac{3}{2}L_{Y^A}T^\rho{}_{[\mu\nu}\phi^A_{,\rho]} = 0 \qquad (35)$$

and finally the scalar field equations

$$g^{\mu\nu}\overset{\circ}{\nabla}_\mu\left(L_{Y^A}T^\rho{}_{\rho\nu} - L_{X^{AB}}\phi^B_{,\nu}\right) - L_{\phi^A} = \vartheta_A\,. \qquad (36)$$

Under a *conformal transformation* (16) the scalar quantities in the action transform as

$$\bar{T} = e^{-2\gamma}\left(T + 4\gamma_{,A}Y^A + 12\gamma_{,A}\gamma_{,B}X^{AB}\right)\,, \quad \bar{\phi}^A = f^A\,,$$
$$\bar{X}^{AB} = e^{-2\gamma}\frac{\partial\bar{\phi}^A}{\partial\phi^C}\frac{\partial\bar{\phi}^B}{\partial\phi^D}X^{CD}\,, \quad \bar{Y}^A = e^{-2\gamma}\frac{\partial\bar{\phi}^A}{\partial\phi^B}\left(Y^B + 6\gamma_{,C}X^{BC}\right)\,, \qquad (37)$$

such that the transformed action $\bar{S}_g$ retains the same form (32).

“Scalar-tensor-like” class of theories [3]

A special subclass of the $L(T, X, Y, \phi)$ theories [2] is given by the *gravitational action*

$$S_g\left[\theta^a, \overset{\bullet}{\omega}{}^a{}_b, \phi^A\right] = \frac{1}{2\kappa^2}\int_M \left[-\mathcal{A}(\phi)T + 2\mathcal{B}_{AB}(\phi)X^{AB} + 2\mathcal{C}_A(\phi)Y^A - 2\kappa^2\mathcal{V}(\phi)\right]\theta\mathrm{d}^4x\,. \qquad (38)$$

For the *matter action*, one chooses a conformal coupling

$$S_m[\theta^a, \phi^A, \chi^I] = S_m^{\mathfrak{J}}\left[e^{\alpha(\phi)}\theta^a, \chi^I\right]\,. \qquad (39)$$

The resulting *field equations* are the symmetric tetrad field equations

$$(\mathcal{A}_{,A} + \mathcal{C}_A)S_{(\mu\nu)}{}^\rho\phi^A_{,\rho} + \mathcal{A}\left(\overset{\circ}{R}_{\mu\nu} - \frac{1}{2}\overset{\circ}{R}g_{\mu\nu}\right) + \left(\frac{1}{2}\mathcal{B}_{AB} - \mathcal{C}_{(A,B)}\right)\phi^A_{,\rho}\phi^B_{,\sigma}g^{\rho\sigma}g_{\mu\nu}$$
$$-\left(\mathcal{B}_{AB} - \mathcal{C}_{(A,B)}\right)\phi^A_{,\mu}\phi^B_{,\nu} + \mathcal{C}_A\left(\overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}_\nu\phi^A - \overset{\circ}{\square}\phi^A g_{\mu\nu}\right) + \kappa^2\mathcal{V}g_{\mu\nu} = \kappa^2\Theta_{\mu\nu}\,, \qquad (40)$$

the antisymmetric tetrad / connection field equations

$$3(\mathcal{A}_{,A} + \mathcal{C}_A)T^\rho{}_{[\mu\nu}\phi^A_{,\rho]} + 2\mathcal{C}_{[A,B]}\phi^A_{,\mu}\phi^B_{,\nu} = 0 \qquad (41)$$

and the scalar field equations

$$\frac{1}{2}\mathcal{A}_{,A}T - \mathcal{B}_{AB}\overset{\circ}{\square}\phi^B - \left(\mathcal{B}_{AB,C} - \frac{1}{2}\mathcal{B}_{BC,A}\right)g^{\mu\nu}\phi^B_{,\mu}\phi^C_{,\nu} + \mathcal{C}_A\overset{\circ}{\nabla}_\mu T_\nu{}^{\nu\mu} + 2\mathcal{C}_{[A,B]}T_\mu{}^{\mu\nu}\phi^B_{,\nu} + \kappa^2\mathcal{V}_{,A} = \kappa^2\alpha_{,A}\Theta\,. \qquad (42)$$

Using the trace of the tetrad equations, the scalar equations can be *debraided* to become

$$\mathcal{A}(\mathcal{A}_{,A} + \mathcal{C}_A)T + (\mathcal{C}_A\mathcal{B}_{BC} - 2\mathcal{A}\mathcal{B}_{AB,C} + \mathcal{A}\mathcal{B}_{BC,A} - 3\mathcal{C}_A\mathcal{C}_{B,C})g^{\mu\nu}\phi^B_{,\mu}\phi^C_{,\nu} - (2\mathcal{A}\mathcal{B}_{AB} + 3\mathcal{C}_A\mathcal{C}_B)\overset{\circ}{\square}\phi^B$$
$$+ \left[4\mathcal{A}\mathcal{C}_{[A,B]} - 2\mathcal{C}_A(\mathcal{A}_{,B} + \mathcal{C}_B)\right]T_\mu{}^{\mu\nu}\phi^B_{,\nu} + 2\kappa^2(\mathcal{A}\mathcal{V}_{,A} + 2\mathcal{C}_A\mathcal{V}) = \kappa^2(2\mathcal{A}\alpha_A + \mathcal{C}_A)\Theta\,. \qquad (43)$$

A *conformal transformation* (16) relates the action (38) and (39) to a new action $\bar{S}_g$ and $\bar{S}_m$ of the same form, where

$$\mathcal{A} = e^{2\gamma}\bar{\mathcal{A}}\,, \quad \mathcal{V} = e^{4\gamma}\bar{\mathcal{V}}\,, \quad \alpha = \bar{\alpha} + \gamma\,,$$
$$\mathcal{B}_{AB} = e^{2\gamma}\left(\bar{\mathcal{B}}_{CD}\frac{\partial\bar{\phi}^C}{\partial\phi^A}\frac{\partial\bar{\phi}^D}{\partial\phi^B} - 6\bar{\mathcal{A}}\gamma_{,A}\gamma_{,B} + 6\bar{\mathcal{C}}_C\frac{\partial\bar{\phi}^C}{\partial\phi^{(A}}\gamma_{,B)}\right)\,, \quad \mathcal{C}_A = e^{2\gamma}\left(\bar{\mathcal{C}}_B\frac{\partial\bar{\phi}^B}{\partial\phi^A} - 2\bar{\mathcal{A}}\gamma_{,A}\right)\,. \qquad (44)$$

One can define the following *invariant quantities*

$$\mathcal{I}_1 = \frac{e^{2\alpha}}{\mathcal{A}}\,, \quad \mathcal{F}_{AB} = \frac{2\mathcal{A}\mathcal{B}_{AB} - 6\mathcal{A}_{,(A}\mathcal{C}_{B)} - 3\mathcal{A}_{,A}\mathcal{A}_{,B}}{4\mathcal{A}^2}\,, \quad \mathcal{H}_A = \frac{\mathcal{C}_A + \mathcal{A}_{,A}}{2\mathcal{A}}\,, \qquad (45a)$$

$$\mathcal{I}_2 = \frac{\mathcal{V}}{\mathcal{A}^2}\,, \quad \mathcal{G}_{AB} = \frac{\mathcal{B}_{AB} - 6\alpha_{,(A}\mathcal{C}_{B)} - 6\alpha_{,A}\alpha_{,B}\mathcal{A}}{2e^{2\alpha}}\,, \quad \mathcal{K}_A = \frac{\mathcal{C}_A + 2\alpha_{,A}\mathcal{A}}{2e^{2\alpha}}\,, \qquad (45b)$$

and finds that under the transformation law (44) they transform covariantly as

$$\bar{\mathcal{I}}_1 = \mathcal{I}_1\,, \quad \bar{\mathcal{F}}_{AB} = \frac{\partial\phi^C}{\partial\bar{\phi}^A}\frac{\partial\phi^D}{\partial\bar{\phi}^B}\mathcal{F}_{CD}\,, \quad \bar{\mathcal{H}}_A = \frac{\partial\phi^B}{\partial\bar{\phi}^A}\mathcal{H}_B\,, \qquad (46a)$$

$$\bar{\mathcal{I}}_2 = \mathcal{I}_2\,, \quad \bar{\mathcal{G}}_{AB} = \frac{\partial\phi^C}{\partial\bar{\phi}^A}\frac{\partial\phi^D}{\partial\bar{\phi}^B}\mathcal{G}_{CD}\,, \quad \bar{\mathcal{K}}_A = \frac{\partial\phi^B}{\partial\bar{\phi}^A}\mathcal{K}_B\,. \qquad (46b)$$

The theory is *minimally coupled* if $\mathcal{K}_A \equiv 0$. A number of *conformal frames* can be defined by selecting particular conformal transformations (16). In the *Jordan frame* defined by

$$\theta^{\mathfrak{J}a} = e^{\gamma^{\mathfrak{J}}(\phi)}\theta^a = e^{\alpha(\phi)}\theta^a\,, \quad \gamma^{\mathfrak{J}}(\phi) = \alpha(\phi) \qquad (47)$$

the transformed parameter functions take the values

$$\mathcal{A}^{\mathfrak{J}} = \frac{1}{\mathcal{I}_1}\,, \quad \mathcal{B}^{\mathfrak{J}}_{AB} = 2\mathcal{G}_{AB}\,, \quad \mathcal{C}^{\mathfrak{J}}_A = 2\mathcal{K}_A\,, \quad \mathcal{V}^{\mathfrak{J}} = \frac{\mathcal{I}_2}{\mathcal{I}_1^2}\,, \quad \alpha^{\mathfrak{J}} = 0\,. \qquad (48)$$

In the *Einstein frame* defined by

$$\theta^{\mathfrak{E}a} = e^{\gamma^{\mathfrak{E}}(\phi)}\theta^a = \sqrt{\mathcal{A}(\phi)}\theta^a\,, \quad \gamma^{\mathfrak{E}}(\phi) = \frac{1}{2}\ln\mathcal{A}(\phi) \qquad (49)$$

the parameter functions are given by

$$\mathcal{A}^{\mathfrak{E}} = 1\,, \quad \mathcal{B}^{\mathfrak{E}}_{AB} = 2\mathcal{F}_{AB}\,, \quad \mathcal{C}^{\mathfrak{E}}_A = 2\mathcal{H}_A\,, \quad \mathcal{V}^{\mathfrak{E}} = \mathcal{I}_2\,, \quad \alpha^{\mathfrak{E}} = \frac{1}{2}\ln\mathcal{I}_1\,. \qquad (50)$$

In the special case $\mathcal{H}_A = \tilde{\mathcal{H}}_{,A}$ there also exists a *debraiding frame* defined such that

$$\gamma^{\mathfrak{D}}_{,A}(\phi) = -\frac{\mathcal{C}_A(\phi)}{2\mathcal{A}(\phi)}\,, \qquad (51)$$

in which the parameter functions satisfy

$$\left(\ln\mathcal{A}^{\mathfrak{D}}\right)_{,A} = 2\mathcal{H}_A\,, \quad \left(\ln\mathcal{B}^{\mathfrak{D}}\right)^A{}_{B,C} = \left[\ln(\mathcal{F} + 3\mathcal{H}\otimes\mathcal{H})\right]^A{}_{B,C} + 2\delta^A_B\mathcal{H}_C\,,$$
$$\mathcal{C}^{\mathfrak{D}}_A = 0\,, \quad \left(\ln\mathcal{V}^{\mathfrak{D}}\right)_{,A} = (\ln\mathcal{I}_2)_A + 4\mathcal{H}_A\,, \quad \alpha^{\mathfrak{D}}{}_{,A} = \mathcal{I}_1\mathcal{K}_A\,. \qquad (52)$$

and the second order terms in the tetrad and scalar field equations (40) and (42) disentangle.