Gauge transformations and Lorentz invariance

A geometric view on teleparallel gravity

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Teleparallel Gravity Workshop-Seminar - 21. 6. 2021

Why use tetrads in observations?

- Every observer can establish local frame of reference at $x \in M$:
  - Four-velocity of observer $\rightsquigarrow$ direction of time component.
  - Clock showing proper time $\rightsquigarrow$ normalization of time component.
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$\Rightarrow$ Need prescription to translate quantities between different frames.

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↯ Inertial frames in general not extendable beyond world line.

What is local Lorentz covariance?

Statement of local Lorentz covariance

Observable, physical quantities are Lorentz covariant, i.e., at every point $x \in M$ of spacetime M the physical quantities Q, Q' measured at x with respect to orthonormal frames θ, θ', which are related to each other by a (proper) Lorentz transformation $\Lambda \in \mathrm{SO}_0(1,3)$, $\theta = \Lambda\theta'$, are related to each other by some representation $\rho : \mathrm{SO}_0(1,3) \to \mathrm{GL}(n)$ of the Lorentz group, $Q = \rho(\Lambda)Q'$.

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Consequence of local Lorentz covariance

Observable, physical fields are described by sections of bundles associated to the orthonormal frame bundle via their corresponding representation ρ, i.e., they are tensor fields.

The Weitzenböck gauge

- Common lore: *One can always use the Weitzenböck gauge.*
  - The spin connection is flat:

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⇒ *The spin connection can always be written in the form*

$$\omega^a{}_{b\mu} = \Lambda^a{}_c \partial_\mu (\Lambda^{-1})^c{}_b\,. \tag{4}$$

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- Remark: this holds also in symmetric and general teleparallelism.

How to obtain the Weitzenböck gauge?

- Recall that we have gauge invariant quantities:
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  - ✓ Global Lorentz invariance encoded in freedom of choice for $\overset{w}{\theta}{}^a{}_\mu(x)$.

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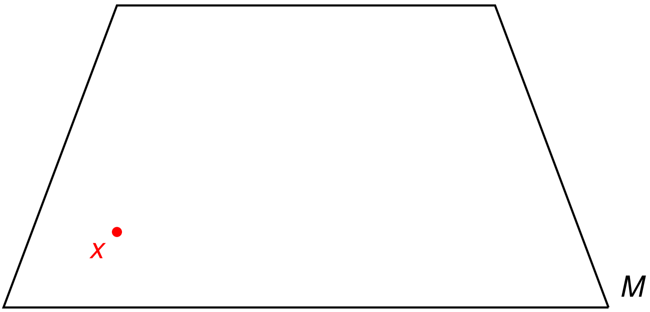

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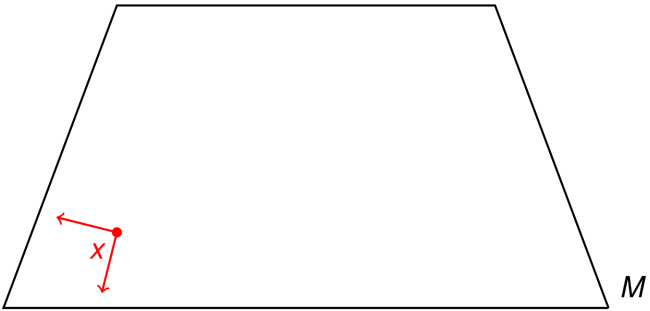

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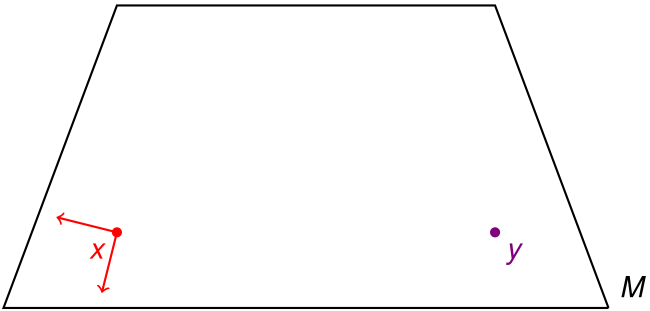

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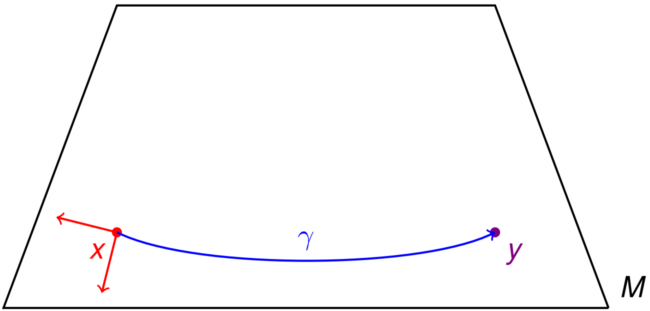

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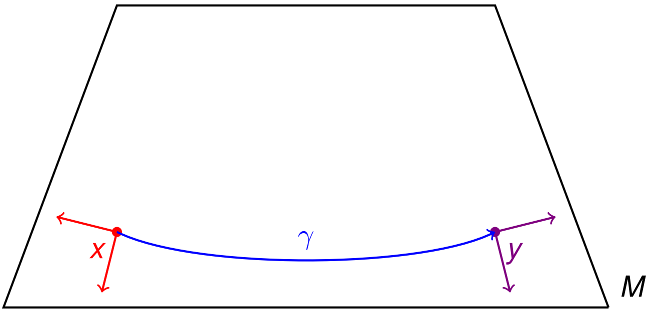

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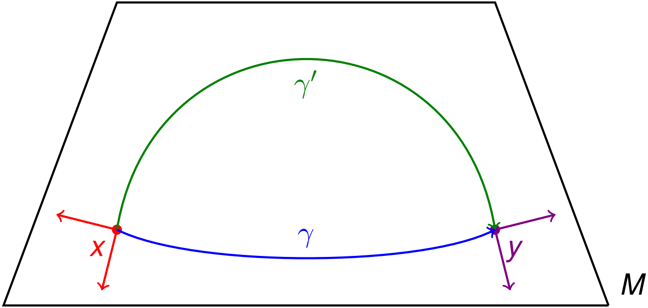
- What happens if we choose another path $x \overset{\gamma'}{\rightsquigarrow} y$?



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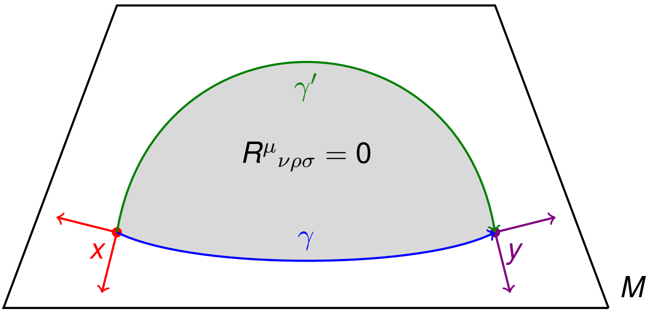
  - ✓ Vanishing curvature: parallel transport along both path agrees.



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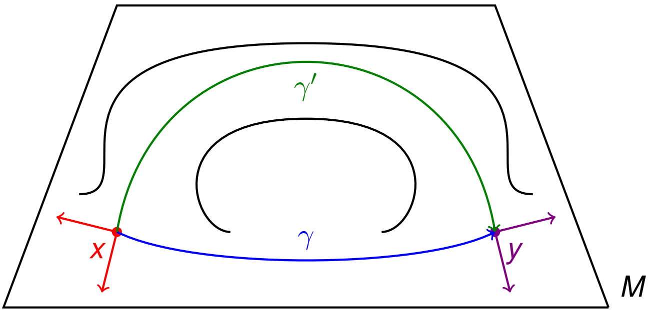
- What happens if we choose another path $x \overset{\gamma'}{\rightsquigarrow} y$?
  - ✓ Vanishing curvature: parallel transport along both path agrees.
  - ↯ But only if γ and γ' are homotopic paths!



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- Consider local Lorentz transformations $\Lambda : M \to \mathrm{SO}(1,3)$:
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- Physical geometry: $\mathrm{SO}_0(1,3)$ reduction of the frame bundle & Γ.

What about the teleparallel affine connection?

- Coupling of the teleparallel affine connection Γ:
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$\Rightarrow$ The "usual rules" for playing with "dark" fields apply:

  - Find out which degrees of freedom couple to physical observables.
  - "Remnant symmetries" may yield gauge degrees of freedom.
  - Make sure physical degrees of freedom obey healthy evolution.
  - ↯ Pay attention to possible pathologies:
    - Is the evolution of physical degrees of freedom determined?
    - Are the physical degrees of freedom stable under perturbations?[1]
    - Does the theory remain healthy under quantization?

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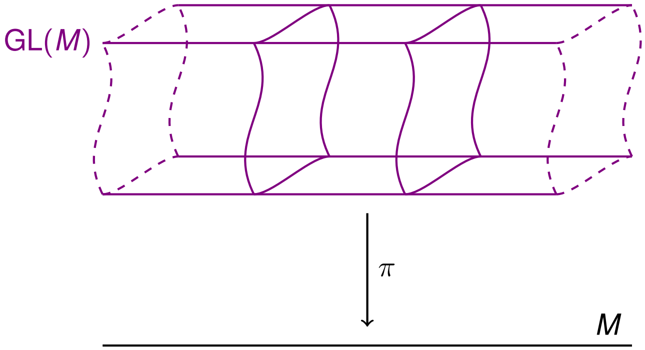
$\Rightarrow$ Most fundamental variables found in geometric picture.

The geometric picture

1. Start with the general linear frame bundle $\pi : \mathrm{GL}(M) \to M$.



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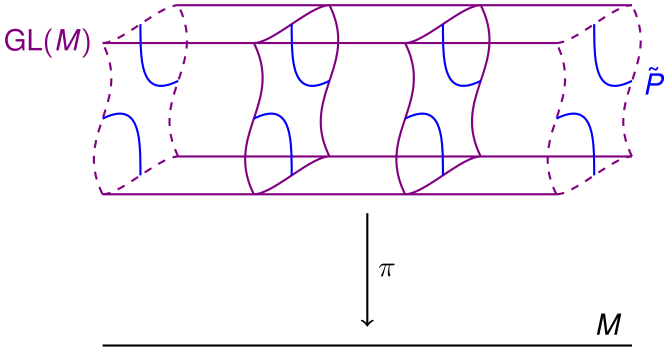

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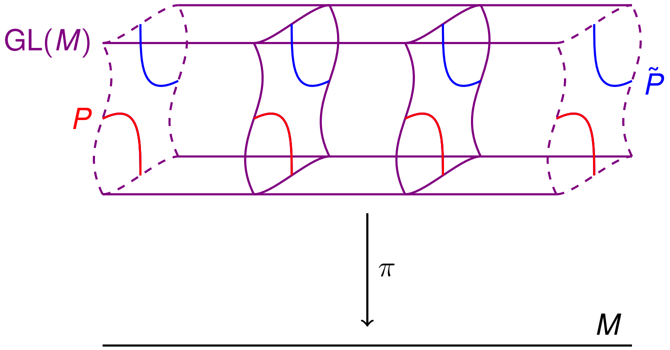

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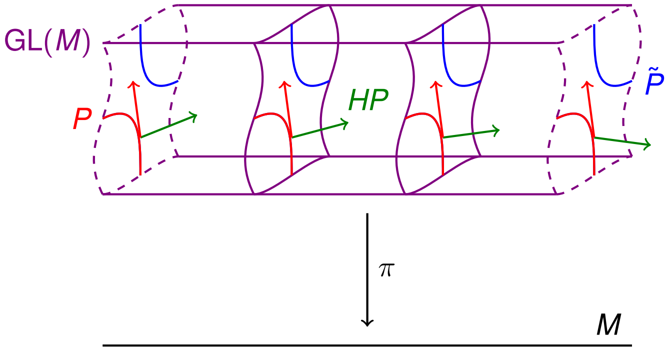

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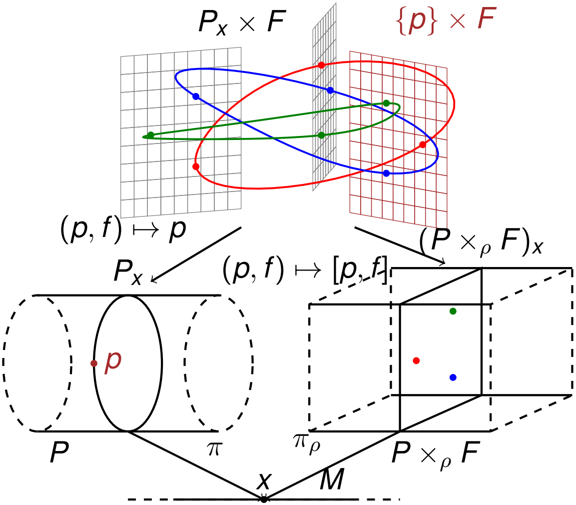
Mantra

In order to understand gravity, one must understand geometry.

Extra: the associated bundle



Extra: the many faces of connections