Biianchi cosmologies in teleparallel and metric-affine gravity

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Motivation

- Open questions in cosmology and gravity:
    - Accelerating phases in the history of the Universe?
    - Relation between gravity and gauge theories?
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- Idea: study geometries with less symmetry:
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  - Bianchi classes II, III, IX admit four-dimensional symmetry groups.
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[1] 125. anniversary of Bianchi classification!

Outline

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Metric-affine and teleparallel geometry

- Fundamental fields in metric-affine geometry:
  - Metric tensor $g_{\mu\nu}$:
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- Metric teleparallelism conventionally formulated using:
  - Tetrad / coframe: $\theta^a = \theta^a{}_{\mu}\mathrm{d}x^{\mu}$ with inverse $e_a = e_a{}^{\mu}\partial_{\mu}$.
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- Induced metric-affine geometry:
  - Metric:

$$g_{\mu\nu} = \eta_{ab}\theta^a{}_{\mu}\theta^b{}_{\nu}\,. \tag{1}$$

  - Affine connection:

$$\Gamma^{\mu}{}_{\nu\rho} = e_a{}^{\mu}\left(\partial_{\rho}\theta^a{}_{\nu} + \omega^a{}_{b\rho}\theta^b{}_{\nu}\right)\,. \tag{2}$$

Symmetries of metric-affine geometries

- Consider action of Lie algebra $\mathfrak{g}$ spanned by X_A with structure constants ${C_{AB}}^C$:

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- Symmetry of metric-affine geometry under action of basis vector fields:
  - Metric:

$$0 = (\mathcal{L}_{X_A} g)_{\mu\nu} = X_A^\rho \partial_\rho g_{\mu\nu} + \partial_\mu X_A^\rho g_{\rho\nu} + \partial_\nu X_A^\rho g_{\mu\rho}\,. \tag{4}$$

  - Connection coefficients:

$$\begin{aligned} 0 = (\mathcal{L}_{X_A}\Gamma)^\mu{}_{\nu\rho} &= X_A^\sigma \partial_\sigma \Gamma^\mu{}_{\nu\rho} - \partial_\sigma X_A^\mu \Gamma^\sigma{}_{\nu\rho} + \partial_\nu X_A^\sigma \Gamma^\mu{}_{\sigma\rho} + \partial_\rho X_A^\sigma \Gamma^\mu{}_{\nu\sigma} + \partial_\nu \partial_\rho X_A^\mu \\ &= \nabla_\rho \nabla_\nu X_A^\mu - X_A^\sigma R^\mu{}_{\nu\rho\sigma} - \nabla_\rho (X_A^\sigma T^\mu{}_{\nu\sigma})\,. \end{aligned} \tag{5}$$

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- Recall Lie algebra property of vector fields:

$$(\mathcal{L}_{[X_A, X_B]} g)_{\mu\nu} = (\mathcal{L}_{X_A} \mathcal{L}_{X_B} g)_{\mu\nu} - (\mathcal{L}_{X_B} \mathcal{L}_{X_A} g)_{\mu\nu} = 0 \,, \tag{6a}$$

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Metric teleparallel geometry and spacetime symmetries

- Express Lie derivative of tetrad in the tetrad basis:

$$(\mathcal{L}_{X_A}\theta)^a{}_\mu = -\boldsymbol{\lambda}_A{}^a{}_b\theta^b{}_\mu\,. \tag{7}$$

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- Metric invariant under X^μ_A if and only if:

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- Affine connection is invariant under X_A^μ if and only if:

$$0 = (\mathcal{L}_{X_A}\omega)^a{}_{b\mu} = \partial_\mu\boldsymbol{\lambda}_A^a{}_b + \omega^a{}_{c\mu}\boldsymbol{\lambda}_A^c{}_b - \omega^c{}_{b\mu}\boldsymbol{\lambda}_A^a{}_c \,. \tag{11}$$

- Express Lie derivative of tetrad in the tetrad basis:

$$(\mathcal{L}_{X_A}\theta)^a{}_\mu = -\boldsymbol{\lambda}^a_{Ab}\theta^b{}_\mu\,. \tag{7}$$

- Recall Lie algebra structure and linearity of Lie derivative:

$$(\mathcal{L}_{[X_A,X_B]}\theta)^a{}_\mu = C_{AB}{}^C(\mathcal{L}_{X_C}\theta)^a{}_\mu\,. \tag{8}$$

⇒ $\boldsymbol{\lambda}^a_{Ab}$ satisfy property of Lie algebra homomorphism:

$$\boldsymbol{\lambda}^a_{Ac}\boldsymbol{\lambda}^c_{Bb} - \boldsymbol{\lambda}^a_{Bc}\boldsymbol{\lambda}^c_{Ab} = C_{AB}{}^C\boldsymbol{\lambda}^a_{Cb}\,. \tag{9}$$

- Metric invariant under X_A^μ if and only if $\boldsymbol{\lambda}_A \in \mathfrak{so}(1,3)$:

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- Use Weitzenböck gauge $\omega^a{}_{b\mu} \equiv 0$: Lie algebra homomorphism $\boldsymbol{:}\ \mathfrak{g} \to \mathfrak{so}(1,3)$.

Global Lorentz transformations

- Geometry in Weitzenböck gauge unchanged under global Lorentz transformation $\Lambda^a{}_b$:

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  3. My brain (no idea how much RAM): solved in less than 24 BPU hours.

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Vector product representation

- Commutation relations for basis elements:

$$[J_i, J_j] = \epsilon_{ijk} J_k \,, \quad [K_i, K_j] = -\epsilon_{ijk} J_k \,, \quad [J_i, K_j] = \epsilon_{ijk} K_k \,. \tag{16}$$

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- Introduce vector notation for $\boldsymbol{\lambda}_A$ with $\vec{j}_A, \vec{k}_A \in \mathbb{R}^3$:

$$\boldsymbol{\lambda}_A = \vec{j}_A \cdot \vec{J} + \vec{k}_A \cdot \vec{K} \,, \quad \vec{J} = (J_1, J_2, J_3) \,, \quad \vec{K} = (K_1, K_2, K_3) \,, \tag{17}$$

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⇒ Commutator of two arbitrary algebra elements:

$$[\vec{j} \cdot \vec{J} + \vec{k} \cdot \vec{K}, \vec{j}' \cdot \vec{J} + \vec{k}' \cdot \vec{K}] = (\vec{j} \times \vec{j}' - \vec{k} \times \vec{k}') \cdot \vec{J} + (\vec{j} \times \vec{k}' + \vec{k} \times \vec{j}') \cdot \vec{K}\,, \tag{19}$$

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⇒ Homomorphism condition for $\boldsymbol{\lambda}$:

$$\vec{j}_A \times \vec{j}_B - \vec{k}_A \times \vec{k}_B = C_{AB}{}^C \vec{j}_C \,, \quad \vec{j}_A \times \vec{k}_B + \vec{k}_A \times \vec{j}_B = C_{AB}{}^C \vec{k}_C \,. \tag{20}$$

Adjoint representation of the Lorentz group

- Adjoint representation of one-parameter groups with unit vector $\vec{n}$:

$$e^{t\vec{n}\cdot\vec{J}}(\vec{j}\cdot\vec{J})e^{-t\vec{n}\cdot\vec{J}} = [(\vec{n}\cdot\vec{j})\vec{n} + \vec{n}\times\vec{j}\sin t - \vec{n}\times(\vec{n}\times\vec{j})\cos t]\cdot\vec{J}\,, \tag{21a}$$

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   a. $D = 0$: parametrized by $u \in \mathbb{R}^+$ as $\|\vec{j}\| = \|\vec{k}\| = u > 0$.
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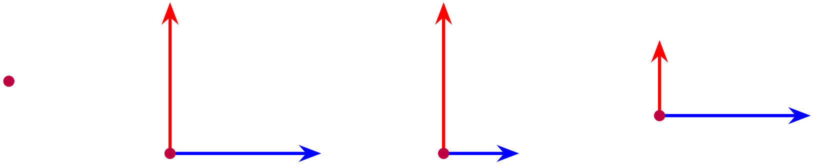

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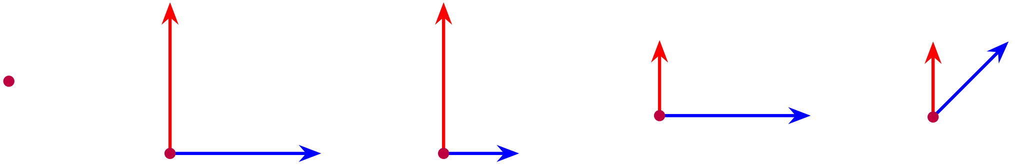
3. One non-orthogonal orbit for each $|S| > 0$ and $D \in \mathbb{R}$:
   - Each orbit parametrized by $u = \cos\sphericalangle(\vec{j}, \vec{k}) \in [-1, 1] \setminus \{0\}$.
   - Elements of the orbits satisfy

$$\|\vec{j}\|^2 = \sqrt{\frac{D^2}{4} + \frac{S^2}{u^2}} + \frac{D}{2}, \quad \|\vec{k}\|^2 = \sqrt{\frac{D^2}{4} + \frac{S^2}{u^2}} - \frac{D}{2}. \tag{25}$$

   - Two connected components $\operatorname{sgn} S = \operatorname{sgn} u = \pm 1$ related by reflections.



Outline

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Free functions in most general geometries

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	no symmetry	only homogeneous	1 isotropy	3 isotropies
Metric	$g_{\mu\nu}$	$\mathcal{G}_{1\ldots10}(t)$	$\mathcal{G}_{1\ldots4}(t)$	$\mathcal{G}_{1\ldots2}(t)$
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Torsion	$T^{\mu}{}_{\nu\rho}$	$\mathcal{T}_{1\ldots24}(t)$	$\mathcal{T}_{1\ldots8}(t)$	$\mathcal{T}_{1\ldots2}(t)$
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- Here: consider Bianchi types II, III, IX:
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  - Metric teleparallel geometry in Weitzenböck gauge: tetrad only.

Outline

Symmetry generators

- Symmetry algebra:

$$[X_1, X_2] = [X_1, X_3] = [X_1, X_4] = 0\,, \quad [X_2, X_3] = X_1\,, \quad [X_2, X_4] = -X_3\,, \quad [X_3, X_4] = X_2\,. \tag{26}$$

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- Algebra represented by vector fields:

$$X_1 = \partial_y\,, \quad X_2 = \partial_z\,, \quad X_3 = z\partial_y - \partial_x\,, \quad X_4 = z\partial_x + \frac{x^2 - z^2}{2}\partial_y - x\partial_z\,. \tag{27}$$

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⇒ Homomorphism conditions:

$$\vec{j}_1 \times \vec{j}_2 - \vec{k}_1 \times \vec{k}_2 = 0\,, \qquad \vec{j}_1 \times \vec{k}_2 + \vec{k}_1 \times \vec{j}_2 = 0\,, \tag{28a}$$

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$$\vec{j}_1 \times \vec{j}_4 - \vec{k}_1 \times \vec{k}_4 = 0\,, \qquad \vec{j}_1 \times \vec{k}_4 + \vec{k}_1 \times \vec{j}_4 = 0\,, \tag{28c}$$

$$\vec{j}_2 \times \vec{j}_3 - \vec{k}_2 \times \vec{k}_3 = \vec{j}_1\,, \qquad \vec{j}_2 \times \vec{k}_3 + \vec{k}_2 \times \vec{j}_3 = \vec{k}_1\,, \tag{28d}$$

$$\vec{j}_2 \times \vec{j}_4 - \vec{k}_2 \times \vec{k}_4 = -\vec{j}_3\,, \qquad \vec{j}_2 \times \vec{k}_4 + \vec{k}_2 \times \vec{j}_4 = -\vec{k}_3\,, \tag{28e}$$

$$\vec{j}_3 \times \vec{j}_4 - \vec{k}_3 \times \vec{k}_4 = \vec{j}_2\,, \qquad \vec{j}_3 \times \vec{k}_4 + \vec{k}_3 \times \vec{j}_4 = \vec{k}_2\,. \tag{28f}$$

1. Trivial branch: $\vec{j}_4, \vec{k}_4$ arbitrary and

$$\vec{j}_1 = \vec{j}_2 = \vec{j}_3 = \vec{k}_1 = \vec{k}_2 = \vec{k}_3 = 0\,. \tag{29}$$

Lie algebra homomorphisms

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2. Non-trivial branch:
   - Conditions to be satisfied:

$$\begin{gathered}\vec{j}_3 \perp \vec{k}_3\,, \quad \|\vec{j}_3\| = \|\vec{k}_3\|\,, \quad \vec{j}_4 \perp \vec{k}_4\,, \quad \|\vec{j}_4\|^2 - \|\vec{k}_4\|^2 = 1\,,\\ \vec{j}_3 \cdot \vec{k}_4 + \vec{j}_4 \cdot \vec{k}_3 = \vec{j}_3 \cdot \vec{j}_4 - \vec{k}_3 \cdot \vec{k}_4 = 0\,, \quad \vec{j}_1 = \vec{k}_1 = 0\,,\\ \vec{j}_3 \times \vec{j}_4 - \vec{k}_3 \times \vec{k}_4 = \vec{j}_2\,, \quad \vec{j}_3 \times \vec{k}_4 + \vec{k}_3 \times \vec{j}_4 = \vec{k}_2\,.\end{gathered} \tag{30}$$

   ⇒ Representative for unique orbit:

$$\vec{j}_4 = \vec{e}_3\,, \quad \vec{k}_3 = \vec{j}_2 = \vec{e}_1\,, \quad \vec{j}_3 = -\vec{k}_2 = \vec{e}_2\,, \quad \vec{j}_1 = \vec{k}_1 = \vec{k}_4 = 0\,. \tag{31}$$

Symmetric teleparallel geometry

1. Trivial homomorphism: no non-degenerate tetrad solutions.

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2. Non-trivial homomorphism:
   - Tetrad:

$$\begin{aligned}\theta^0 &= [\mathcal{C}_2(1+x^2+z^2)+\mathcal{C}_1]\mathrm{d}t + (\mathcal{C}_3x+\mathcal{C}_4z)\mathrm{d}x + [\mathcal{C}_6(1+x^2+z^2)+\mathcal{C}_5]\mathrm{d}y \\ &\quad + \{[\mathcal{C}_6(1+x^2+z^2)+\mathcal{C}_5-\mathcal{C}_4]x+\mathcal{C}_3z\}\mathrm{d}z\,, &\text{(32a)}\\ \theta^1 &= 2\mathcal{C}_2x\mathrm{d}t + \mathcal{C}_3\mathrm{d}x + 2\mathcal{C}_6x\mathrm{d}y + (2\mathcal{C}_6x^2-\mathcal{C}_4)\mathrm{d}z\,, &\text{(32b)}\\ \theta^2 &= 2\mathcal{C}_2z\mathrm{d}t + \mathcal{C}_4\mathrm{d}x + 2\mathcal{C}_6z\mathrm{d}y + (2\mathcal{C}_6xz+\mathcal{C}_3)\mathrm{d}z\,, &\text{(32c)}\\ \theta^3 &= [\mathcal{C}_2(1-x^2-z^2)-\mathcal{C}_1]\mathrm{d}t - (\mathcal{C}_3x+\mathcal{C}_4z)\mathrm{d}x + [\mathcal{C}_6(1-x^2-z^2)-\mathcal{C}_5]\mathrm{d}y \\ &\quad + \{[\mathcal{C}_6(1-x^2-z^2)-\mathcal{C}_5+\mathcal{C}_4]x-\mathcal{C}_3z\}\mathrm{d}z\,, &\text{(32d)}\end{aligned}$$

   - Determinant:

$$\det\theta = 2(\mathcal{C}_2\mathcal{C}_5-\mathcal{C}_1\mathcal{C}_6)(\mathcal{C}_3^2+\mathcal{C}_4^2)\,. \tag{33}$$

   - Metric:

$$\begin{gathered} g_{tt} = -4\mathcal{C}_1\mathcal{C}_2\,, \quad g_{ty} = -2(\mathcal{C}_2\mathcal{C}_5+\mathcal{C}_1\mathcal{C}_6)\,, \quad g_{tz} = xg_{ty}\,, \quad g_{xx} = \mathcal{C}_3^2+\mathcal{C}_4^2\,, \\ g_{yy} = -4\mathcal{C}_5\mathcal{C}_6\,, \quad g_{yz} = xg_{yy}\,, \quad g_{zz} = \mathcal{C}_3^2+\mathcal{C}_4^2-4\mathcal{C}_5\mathcal{C}_6x^2\,. \end{gathered} \tag{34}$$

Outline

Symmetry generators

- Symmetry algebra:

$$[X_1, X_2] = [X_2, X_3] = [X_2, X_4] = 0\,, \quad [X_1, X_3] = -X_1\,, \quad [X_1, X_4] = X_3\,, \quad [X_3, X_4] = -X_4\,. \tag{35}$$

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- Algebra represented by vector fields ($0 < n < 1$):

$$X_1 = \partial_y\,, \quad X_2 = \partial_z\,, \quad X_3 = \partial_x - y\partial_y\,, \quad X_4 = y\partial_x + \frac{1}{2}\left(\frac{e^{-2x}}{1-n^2} - y^2\right)\partial_y - \frac{ne^{-x}}{1-n^2}\partial_z\,, \tag{36}$$

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⇒ Homomorphism conditions:

$$\vec{j}_1 \times \vec{j}_2 - \vec{k}_1 \times \vec{k}_2 = 0\,, \qquad \vec{j}_1 \times \vec{k}_2 + \vec{k}_1 \times \vec{j}_2 = 0\,, \tag{37a}$$

$$\vec{j}_1 \times \vec{j}_3 - \vec{k}_1 \times \vec{k}_3 = -\vec{j}_1\,, \qquad \vec{j}_1 \times \vec{k}_3 + \vec{k}_1 \times \vec{j}_3 = -\vec{k}_1\,, \tag{37b}$$

$$\vec{j}_1 \times \vec{j}_4 - \vec{k}_1 \times \vec{k}_4 = \vec{j}_3\,, \qquad \vec{j}_1 \times \vec{k}_4 + \vec{k}_1 \times \vec{j}_4 = \vec{k}_3\,, \tag{37c}$$

$$\vec{j}_2 \times \vec{j}_3 - \vec{k}_2 \times \vec{k}_3 = 0\,, \qquad \vec{j}_2 \times \vec{k}_3 + \vec{k}_2 \times \vec{j}_3 = 0\,, \tag{37d}$$

$$\vec{j}_2 \times \vec{j}_4 - \vec{k}_2 \times \vec{k}_4 = 0\,, \qquad \vec{j}_2 \times \vec{k}_4 + \vec{k}_2 \times \vec{j}_4 = 0\,, \tag{37e}$$

$$\vec{j}_3 \times \vec{j}_4 - \vec{k}_3 \times \vec{k}_4 = -\vec{j}_4\,, \qquad \vec{j}_3 \times \vec{k}_4 + \vec{k}_3 \times \vec{j}_4 = -\vec{k}_4\,. \tag{37f}$$

Lie algebra homomorphisms

1. Trivial branch: $\vec{j}_2, \vec{k}_2$ arbitrary and

$$\vec{j}_1 = \vec{j}_3 = \vec{j}_4 = \vec{k}_1 = \vec{k}_3 = \vec{k}_4 = 0\,. \tag{38}$$

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2. Non-trivial branch:
   - Conditions to be satisfied:

$$\begin{gathered}\vec{j}_1 \perp \vec{k}_1\,, \quad \|\vec{j}_1\| = \|\vec{k}_1\|\,, \quad \vec{j}_4 \perp \vec{k}_4\,, \quad \|\vec{j}_4\| = \|\vec{k}_4\|\,, \\ \vec{j}_1 \cdot \vec{k}_4 + \vec{j}_4 \cdot \vec{k}_1 = \vec{j}_1 \cdot \vec{j}_4 - \vec{k}_1 \cdot \vec{k}_4 + 1 = 0\,, \quad \vec{j}_2 = \vec{k}_2 = 0\,, \\ \vec{j}_1 \times \vec{j}_4 - \vec{k}_1 \times \vec{k}_4 = \vec{j}_3\,, \quad \vec{j}_1 \times \vec{k}_4 + \vec{k}_1 \times \vec{j}_4 = \vec{k}_3\,. \end{gathered} \tag{39}$$

   $\Rightarrow$ Representative for unique orbit:

$$\vec{k}_3 = \vec{e}_3\,, \quad \vec{j}_1 = -\vec{j}_4 = \frac{1}{\sqrt{2}}\vec{e}_1\,, \quad \vec{k}_1 = \vec{k}_4 = \frac{1}{\sqrt{2}}\vec{e}_2\,, \quad \vec{j}_2 = \vec{j}_3 = \vec{k}_2 = 0\,. \tag{40}$$

Symmetric teleparallel geometry

1. Trivial homomorphism: no non-degenerate tetrad solutions.

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2. Non-trivial homomorphism (use abbreviation $N = \sqrt{1 - n^2}$):
   - Tetrad:

$$\begin{aligned}
\theta^{0,3} &= \frac{\mathcal{C}_1}{2\sqrt{2}}\left[\frac{e^{-x}}{N} + N(y^2 \pm 2)e^x\right]\mathrm{d}t + \frac{\mathcal{C}_4}{2\sqrt{2}n}\left[\frac{e^x}{N} + N(y^2 \pm 2)e^x\right]\mathrm{d}z \\
&\quad + \left\{\left[\frac{e^{-x}}{N} - (y^2 \pm 2)e^x\right]\frac{\mathcal{C}_2}{2\sqrt{2}} - \frac{\mathcal{C}_3 y}{\sqrt{2}N}\right\}\mathrm{d}x \\
&\quad + \frac{1}{2\sqrt{2}}\left[N(\mathcal{C}_3 + \mathcal{C}_4)(y^2 \pm 2)e^{2x} - \frac{\mathcal{C}_3 - \mathcal{C}_4}{N} - 2ye^x\mathcal{C}_2\right]\mathrm{d}y\,, \qquad (41a) \\
\theta^1 &= \mathcal{C}_5\mathrm{d}t + \mathcal{C}_6 e^x\mathrm{d}y + \frac{\mathcal{C}_6}{n}\mathrm{d}z\,, \qquad (41b) \\
\theta^2 &= -N\mathcal{C}_1 ye^x\mathrm{d}t + \left(\mathcal{C}_2 ye^x + \frac{\mathcal{C}_3}{N}\right)\mathrm{d}x + \left[\mathcal{C}_2 e^x - N(\mathcal{C}_3 + \mathcal{C}_4)ye^{2x}\right]\mathrm{d}y - \frac{N}{n}\mathcal{C}_4 ye^x\mathrm{d}z\,. \qquad (41c)
\end{aligned}$$

   - Determinant:

$$\det\theta = (\mathcal{C}_2^2 + \mathcal{C}_3^2)(\mathcal{C}_4\mathcal{C}_5 - \mathcal{C}_1\mathcal{C}_6)\frac{e^x}{nN}\,. \qquad (42)$$

   - Metric:

$$g_{tt} = \mathcal{C}_5^2 - \mathcal{C}_1^2\,, \quad g_{tz} = \frac{\mathcal{C}_5\mathcal{C}_6 - \mathcal{C}_1\mathcal{C}_4}{n}\,, \quad g_{ty} = ng_{tz}e^x\,, \quad g_{xx} = \frac{\mathcal{C}_2^2 + \mathcal{C}_3^2}{N^2}\,,$$
$$g_{zz} = \frac{\mathcal{C}_6^2 - \mathcal{C}_4^2}{n^2}\,, \quad g_{yz} = ng_{zz}e^x\,, \quad g_{yy} = (\mathcal{C}_2^2 + \mathcal{C}_3^2 - \mathcal{C}_4^2 + \mathcal{C}_6^2)e^{2x}\,. \qquad (43)$$

Outline

Symmetry generators

- Symmetry algebra:

$$[X_1, X_2] = X_3\,, \quad [X_2, X_3] = X_1\,, \quad [X_3, X_1] = X_2\,, \quad [X_1, X_4] = [X_2, X_4] = [X_3, X_4] = 0\,. \tag{44}$$

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- Algebra represented by vector fields ($n \neq 0$):

$$\begin{aligned} X_1 &= \partial_y\,, \quad X_2 = n\frac{\sin y}{\sin x}\partial_z + \cos y\partial_x - \frac{\sin y}{\tan x}\partial_y\,, \\ X_3 &= n\frac{\cos y}{\sin x}\partial_z - \sin y\partial_x - \frac{\cos y}{\tan x}\partial_y\,, \quad X_4 = \partial_z\,, \end{aligned} \tag{45}$$

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⇒ Homomorphism conditions:

$$\vec{j}_2 \times \vec{j}_3 - \vec{k}_2 \times \vec{k}_3 = \vec{j}_1\,, \qquad \vec{j}_2 \times \vec{k}_3 + \vec{k}_2 \times \vec{j}_3 = \vec{k}_1\,, \tag{46a}$$

$$\vec{j}_3 \times \vec{j}_1 - \vec{k}_3 \times \vec{k}_1 = \vec{j}_2\,, \qquad \vec{j}_3 \times \vec{k}_1 + \vec{k}_3 \times \vec{j}_1 = \vec{k}_2\,, \tag{46b}$$

$$\vec{j}_1 \times \vec{j}_2 - \vec{k}_1 \times \vec{k}_2 = \vec{j}_3\,, \qquad \vec{j}_1 \times \vec{k}_2 + \vec{k}_1 \times \vec{j}_2 = \vec{k}_3\,, \tag{46c}$$

$$\vec{j}_1 \times \vec{j}_4 - \vec{k}_1 \times \vec{k}_4 = 0\,, \qquad \vec{j}_2 \times \vec{k}_4 + \vec{k}_2 \times \vec{j}_4 = 0\,, \tag{46d}$$

$$\vec{j}_2 \times \vec{j}_4 - \vec{k}_2 \times \vec{k}_4 = 0\,, \qquad \vec{j}_1 \times \vec{k}_4 + \vec{k}_1 \times \vec{j}_4 = 0\,, \tag{46e}$$

$$\vec{j}_3 \times \vec{j}_4 - \vec{k}_3 \times \vec{k}_4 = 0\,, \qquad \vec{j}_3 \times \vec{k}_4 + \vec{k}_3 \times \vec{j}_4 = 0\,. \tag{46f}$$

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2. Non-trivial branch:
   - Conditions to be satisfied:

$$\begin{gathered}\vec{j}_2 \cdot \vec{k}_3 + \vec{j}_3 \cdot \vec{k}_2 = 0\,, \quad \vec{j}_2 \cdot \vec{j}_3 - \vec{k}_2 \cdot \vec{k}_3 = 0\,, \quad \|\vec{j}_2\|^2 - \|\vec{k}_2\|^2 = \|\vec{j}_3\|^2 - \|\vec{k}_3\|^2 = 1\,, \\ \vec{j}_2 \perp \vec{k}_2\,, \quad \vec{j}_3 \perp \vec{k}_3\,, \quad \vec{j}_2 \times \vec{j}_3 - \vec{k}_2 \times \vec{k}_3 = \vec{j}_1\,, \quad \vec{j}_2 \times \vec{k}_3 + \vec{k}_2 \times \vec{j}_3 = \vec{k}_1\,. \end{gathered} \tag{48}$$

   ⇒ Representative for unique orbit:

$$\vec{j}_1 = \vec{e}_1\,, \quad \vec{j}_2 = \vec{e}_2\,, \quad \vec{j}_3 = \vec{e}_3\,, \quad \vec{j}_4 = \vec{k}_1 = \vec{k}_2 = \vec{k}_3 = \vec{k}_4 = 0\,. \tag{49}$$

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$$\theta^0 = \mathcal{C}_1 \mathrm{d}t + n\mathcal{C}_2 \cos x \mathrm{d}y + \mathcal{C}_2 \mathrm{d}z\,, \tag{50a}$$

$$\theta^1 = \mathcal{C}_3 \cos x \mathrm{d}t - \mathcal{C}_4 \sin x \mathrm{d}x + (n\mathcal{C}_5 \cos^2 x - \mathcal{C}_6 \sin^2 x)\mathrm{d}y + \mathcal{C}_5 \cos x \mathrm{d}z\,, \tag{50b}$$

$$\theta^2 = \mathcal{C}_3 \sin x \sin y \mathrm{d}t + (\mathcal{C}_4 \cos x \sin y - \mathcal{C}_6 \cos y)\mathrm{d}x + \mathcal{C}_5 \sin x \sin y \mathrm{d}z + [(n\mathcal{C}_5 + \mathcal{C}_6)\cos x \sin y + \mathcal{C}_4 \cos y] \sin x \mathrm{d}y\,, \tag{50c}$$

$$\theta^3 = \mathcal{C}_3 \sin x \cos y \mathrm{d}t + (\mathcal{C}_4 \cos x \cos y + \mathcal{C}_6 \sin y)\mathrm{d}x + \mathcal{C}_5 \sin x \cos y \mathrm{d}z + [(n\mathcal{C}_5 + \mathcal{C}_6)\cos x \cos y - \mathcal{C}_4 \sin y] \sin x \mathrm{d}y\,, \tag{50d}$$

   - Determinant:

$$\det\theta = (\mathcal{C}_2\mathcal{C}_3 - \mathcal{C}_1\mathcal{C}_5)(\mathcal{C}_4^2 + \mathcal{C}_6^2)\sin x\,, \tag{51}$$

   - Metric:

$$g_{tt} = \mathcal{C}_3^2 - \mathcal{C}_1^2\,, \quad g_{tz} = \mathcal{C}_3\mathcal{C}_5 - \mathcal{C}_1\mathcal{C}_2\,, \quad g_{ty} = n g_{tz} \cos x\,, \quad g_{xx} = \mathcal{C}_4^2 + \mathcal{C}_6^2\,, \quad g_{zz} = \mathcal{C}_5^2 - \mathcal{C}_2^2\,, \quad g_{yz} = n g_{zz} \cos x\,, \quad g_{yy} = g_{xx} \sin^2 x + n^2 g_{zz} \cos^2 x\,. \tag{52}$$

Outline

Conclusion

- Idea:
  - Study metric-affine and teleparallel geometries with spacetime symmetry algebra $\mathfrak{g}$.
  - Most general geometry derived from Lie algebra homomorphism $\boldsymbol{\lambda} : \mathfrak{g} \to \mathfrak{so}(1,3)$.
  - Equivalent geometries related by adjoint representation of Lorentz transformation.
  - Simple vector product representation to determine most general homomorphisms.
  - Application to Bianchi II, III, IX with four-dimensional symmetry algebras.

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- Reference:

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