Dynamical dark energy and cosmological attractors

Dynamical systems approach and cosmological attractors in newer general relativity [arXiv:2505.16917]

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- Open questions in cosmology and gravity:
  - Accelerating phases in the history of the Universe - dark energy, inflation?
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- Modified gravity theories based on symmetric teleparallel gravity:
  - Contains $f(Q)$ gravity [Beltran Jimenez, Heisenberg, Koivisto '17].
  - Contains newer GR [Beltran Jimenez, Heisenberg, Koivisto '17].
  - Contains scalar-non-metricity gravity [Järv, Rünkla, Saal, Vilson '18].

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- Symmetric teleparallel cosmology:
  - Make use of cosmological symmetry in order to find general geometry.
  - Modified Friedmann equations for symmetric teleparallel cosmology.
  - Use method of dynamical systems to study cosmological dynamics.

Symmetric teleparallel gravity

- Fundamental fields in metric-affine geometry:
  - Metric tensor $g_{\mu\nu}$:
    - Defines length of and angle between tangent vectors.
    - Defines length of curves and proper time.
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    - ⋆ Defines parallel transport along arbitrary curves.
    - ⋆ Defines autoparallel curves via parallel transport of tangent vector.

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- Three characteristic quantities:
  - Curvature:

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\tau\rho}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\tau\sigma}\Gamma^{\tau}{}_{\nu\rho} \,. \tag{1}$$

  - Torsion:

$$T^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\rho\nu} - \Gamma^{\mu}{}_{\nu\rho} \,. \tag{2}$$

  - Nonmetricity:

$$Q_{\mu\nu\rho} = \nabla_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \Gamma^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \Gamma^{\sigma}{}_{\rho\mu}g_{\nu\sigma} \,. \tag{3}$$

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- Symmetric teleparallel gravity: curvature and torsion imposed to vanish.

Two families of "newer general relativity" theories

- Newer GR action depends on five parameters $c_1, \ldots, c_5$ [Beltran Jimenez, Heisenberg, Koivisto '17]:

$$S_g = \frac{1}{2\kappa^2} \int_M \left(c_1 Q^{\rho\mu\nu} Q_{\rho\mu\nu} + c_2 Q^{\rho\mu\nu} Q_{\nu\mu\rho} + c_3 Q^\mu Q_\mu + c_4 \tilde{Q}^\mu \tilde{Q}_\mu + c_5 \tilde{Q}^\mu Q_\mu \right) \sqrt{-g}\, d^4x \,. \tag{4}$$

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- Parameters restricted by physical viability conditions:
  - Post-Newtonian limit agrees with GR for two families (2 / 4 parameters) [Flathmann, MH ’21].
  - Absence of ghosts restricts 4-parameter family to 2 parameters [Bello-Morales *et.al.* ’24].
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⇒ Two one-parameter families of viable gravity theories [MH, Karanasou ’25]:
  - Type 1:

$$c_1 = -\frac{1}{4}\,, \quad c_2 = \frac{\epsilon}{4} + \frac{1}{2}\,, \quad c_3 = \frac{1}{4}\,, \quad c_4 = -\frac{\epsilon}{4}\,, \quad c_5 = -\frac{1}{2}\,. \tag{5}$$

  - Type 2:

$$c_1 = -\frac{1}{4}\,, \quad c_2 = \frac{1}{2}\,, \quad c_3 = \frac{\epsilon}{2} + \frac{1}{4}\,, \quad c_4 = 0\,, \quad c_5 = -\frac{1}{2}\,. \tag{6}$$

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- Common limit $\epsilon \to 0$ is symmetric teleparallel equivalent of GR (STEGR):

$$c_1 = -\frac{1}{4}\,, \quad c_2 = \frac{1}{2}\,, \quad c_3 = \frac{1}{4}\,, \quad c_4 = 0\,, \quad c_5 = -\frac{1}{2}\,. \tag{7}$$

Cosmology in newer general relativity

- Impose homogeneity and isotropy on dynamical field variables:
  - ⇒ Metric $g_{\mu\nu}$ takes well-known FLRW form with Hubble parameter H.
  - ⇒ Symmetric teleparallel connection $\Gamma^{\mu}{}_{\nu\rho}$ with free function K and 4 branches [MH '21]:
    - ⋆ One branch for non-zero spatial curvature $k \neq 0$.
    - ⋆ Three spatially flat branches arising as different possibilities to take limit $k \to 0$.

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⇒ Only four combinations of theory parameters appear in dynamical equations [MH '21]:

$$a_1 = 2(c_1 + 3c_3)\,, \tag{8a}$$

$$a_2 = 2(2c_3 + c_5)\,, \tag{8b}$$

$$a_3 = 2(c_1 + c_2 + c_3 + c_4 + c_5)\,, \tag{8c}$$

$$a_4 = 2(c_2 - c_4 + c_5)\,. \tag{8d}$$

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⇒ General structure of cosmological field equations:

- ∘ Branch 1: linear in $\dot{H}, \dot{K}$ and quadratic in H, K.
- ∘ Branch 2 & 3: linear in $\dot{H}, \ddot{K}/K$ and quadratic in $H, K, \dot{K}/K$.

Dynamical systems: degenerate vs non-degenerate

- Transform branches 2 & 3 into first order equations:
  - Introduce new dynamical variable $L = \dot{K}/K$.

$\Rightarrow$ Eliminate time derivatives of K using new dynamical equation $\dot{K} = KL$:

$$\dot{K} = KL\,, \quad \ddot{K} = \dot{K}L + K\dot{L} = K(L^2 + \dot{L})\,. \tag{9}$$

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- Non-degenerate system for $4a_1a_3 - 3a_2^2 \neq 0$:
  - System can be solved for $\dot{H}, \dot{K}$ (branch 1) or $\dot{H}, \dot{K}, \dot{L}$ (branch 2 & 3).
  - Two- or three-dimensional dynamical system.

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  - Two- or three-dimensional dynamical system.
- Degenerate system for $4a_1a_3 - 3a_2^2 = 0$:
  - All time derivatives can be eliminated by at least linear combination of equations.
  - $\Rightarrow$ Cosmological phase space restricted by constraint equation(s).
  - $\Rightarrow$ Solve constraint and parametrize solutions with new variables S, T.
  - $\Rightarrow$ Dynamical system in new variables S, T.

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- Decomposition of variables into angular part (unit vector $\underline{n}$) and radial part (length Z):

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$$\dot{Z} = Z^2 \underline{f}(\underline{n}, \underline{n}) \cdot \underline{n} \,, \tag{12a}$$

$$\dot{\underline{n}} = Z \left\{ \underline{f}(\underline{n}, \underline{n}) - [\underline{f}(\underline{n}, \underline{n}) \cdot \underline{n}]\, \underline{n} \right\} \,. \tag{12b}$$

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⇒ Qualitative dynamics (up to positive factor Z) fully determined by $\underline{n}$.

Fixed points and projective fixed points

- Fixed points: $\dot{\underline{z}} = 0$.
  - $\Rightarrow$ $\underline{z} = 0$ is always a non-hyperbolic fixed point (saddle point).
  - $\Rightarrow$ If $\underline{z} = \underline{z}^*$ is a fixed point, then all $c\underline{z}^*$ with $c \in \mathbb{R}$ are fixed points.
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  - $\Rightarrow$ $\underline{z}^*$ and $-\underline{z}^*$ have opposite stability properties (eigenvalues of Jacobian).
- Projective fixed points: $\dot{\underline{n}} = 0$.
  - $\Rightarrow$ Condition depends only on angular coordinates:

$$\frac{\dot{\underline{n}}}{Z} = \underline{f}(\underline{n}, \underline{n}) - [\underline{f}(\underline{n}, \underline{n}) \cdot \underline{n}]\, \underline{n} = 0\,. \tag{13}$$

  - $\Rightarrow$ If $\underline{n} = \underline{n}^\star$ is a projective fixed point, then $-\underline{n}^\star$ is a projective fixed point.
  - $\Rightarrow$ $\underline{n}^\star$ and $-\underline{n}^\star$ have opposite stability properties.
  - $\Rightarrow$ Since $\dot{\underline{n}} = 0$, $N^\star = \underline{f}(\underline{n}^\star, \underline{n}^\star) \cdot \underline{n}^\star$ is constant at a projective fixed point.
  - $\Rightarrow$ Radial dynamics $\dot{Z} = N^\star Z^2$ can be solved at projective fixed point:

$$Z(t) = \frac{1}{N^\star(t_0 - t)}\,. \tag{14}$$

Phenomenological properties

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- Effective dark energy:
  - Write cosmological dynamics as modified Friedmann equation:

$$3H^2 = \kappa^2 \rho_\lambda \,, \quad -2\dot{H} - 3H^2 = \kappa^2 p_\lambda \,. \tag{15}$$

⇒ Effective barotropic index:

$$w_\lambda = \frac{p_\lambda}{\rho_\lambda} = -1 - \frac{2\dot{H}}{3H^2} \,. \tag{16}$$

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  - $\dot{H}$ depends on phase space coordinates $\Rightarrow$ dark energy becomes dynamical.
- Asymptotic behavior and finite time singularities:
  - Asymptotic behavior determined by projective fixed points (repellers and attractors).
  - Radial dynamics $\dot{Z}$ and direction $\underline{n}$ determine asymptotic Hubble parameter.

Cosmology in type 1 theories

- Parameters in cosmological field equations - degenerate system:

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⇒ Dynamics obtained by solving constrained system:

- Branch 1 becomes identical to STEGR with trivial vacuum cosmology.
- Branch 2: decelerating expansion after past time singularity with $w_\lambda = \frac{1}{9}$:

$$H(t) = \frac{3}{5(t - t_0)}\,, \tag{18}$$

- Branch 3: decelerating expansion after past time singularity with $w_\lambda = 1$:

$$H(t) = \frac{1}{3(t - t_0)}\,, \tag{19}$$

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⇒ No dark energy.

Type 2 theories: dynamical system

- Parameters in cosmological field equations - non-degenerate system:

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⇒ Dynamics obtained by solving for time derivatives $\dot{H}, \dot{K}, \dot{L}$.

- Branch 1:

$$\dot{H} = \epsilon(3H + K)^2\,, \tag{21a}$$

$$\dot{K} = -\frac{1}{2}(3H + K)(9H + K) - 3\epsilon(3H + K)^2 - \frac{3H^2}{2\epsilon}\,. \tag{21b}$$

- Branch 2:

$$\dot{H} = \epsilon(5H + L)^2\,, \tag{22a}$$

$$\dot{K} = KL\,, \tag{22b}$$

$$\dot{L} = -\frac{1}{2}(5H + L)(11H + L) - 5\epsilon(5H + L)^2 - \frac{3H^2}{2\epsilon}\,. \tag{22c}$$

- Branch 3:

$$\dot{H} = \epsilon(3H - 4K - L)^2\,, \tag{23a}$$

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$$\dot{L} = \frac{27}{2}H^2 - 6H(4K + L) + 8K^2 + \frac{1}{2}L^2 + 3\epsilon(3H - 4K - L)^2 + \frac{3H^2}{2\epsilon}\,. \tag{23c}$$

Type 2 theories: cosmological phenomenology

- Hubble parameter: $\dot{H} = \epsilon(xH + yK + zL)^2$ for all three branches.
  - $\Rightarrow$ For $\epsilon > 0$: $\dot{H} \geq 0$, only bounces are possible, no turnarounds.
  - $\Rightarrow$ For $\epsilon < 0$: $\dot{H} \leq 0$, only turnarounds are possible, no bounces.

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- Effective dark energy:
  - General form of barotropic index:

$$w_\lambda = -1 - \frac{2\epsilon}{3H^2}(xH + yK + zL)^2 \,. \tag{24}$$

  - $\Rightarrow$ $w_\lambda \leq -1$ for $\epsilon > 0$ and $w_\lambda \geq -1$ for $\epsilon < 0$.
  - $\Rightarrow$ No crossing of phantom divide possible.

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$$w_\lambda = -1 - \frac{2\epsilon}{3H^2}(xH + yK + zL)^2 \,. \tag{24}$$

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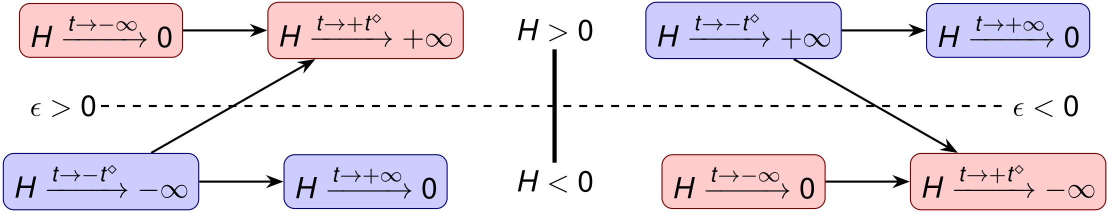
  - $\Rightarrow$ No crossing of phantom divide possible.
- Asymptotic behavior and finite time singularities with $\dot{Z} > 0$ or $\dot{Z} < 0$:



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- Outlook:
  - Include matter (single barotropic fluid / dust + radiation) to get full dynamics.
  - Study cosmological perturbations.
  - Possible generalization to Bianchi spacetime models.
  - Consider general teleparallel gravity with torsion and nonmetricity.