Projective coordinate transformation in teleparallel cosmology

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- Open questions in cosmology and gravity:
  - Accelerating phases in the history of the Universe - dark energy, inflation?
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  - *Metric* teleparallel gravity: only torsion.
  - *Symmetric* teleparallel gravity: only nonmetricity.
  - *General* teleparallel gravity: torsion and nonmetricity.

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  - *General* teleparallel gravity: torsion and nonmetricity.
- Classes of teleparallel cosmology:
  - Make use of cosmological symmetry in order to find general geometry.
  - Modified Friedmann equations for symmetric teleparallel cosmology.
  - Use method of dynamical systems to study cosmological dynamics.

Introduction to teleparallel gravity

- Fundamental fields in metric-affine geometry:
  - Metric tensor $g_{\mu\nu}$:
    - Defines length of and angle between tangent vectors.
    - Defines length of curves and proper time.
    - Defines causality (spacelike and timelike directions).

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  - Connection with coefficients $\Gamma^{\mu}{}_{\nu\rho}$:
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- Three characteristic quantities:
  - Curvature:

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\tau\rho}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\tau\sigma}\Gamma^{\tau}{}_{\nu\rho} \,. \tag{1}$$

  - Torsion:

$$T^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\rho\nu} - \Gamma^{\mu}{}_{\nu\rho} \,. \tag{2}$$

  - Nonmetricity:

$$Q_{\mu\nu\rho} = \nabla_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \Gamma^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \Gamma^{\sigma}{}_{\rho\mu}g_{\nu\sigma} \,. \tag{3}$$

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- Teleparallel gravity: curvature imposed to vanish.

Decomposition of the connection

- Affine connection can be decomposed:

$$\Gamma^{\mu}{}_{\nu\rho} = \overset{\circ}{\Gamma}{}^{\mu}{}_{\nu\rho} + M^{\mu}{}_{\nu\rho} = \overset{\circ}{\Gamma}{}^{\mu}{}_{\nu\rho} + K^{\mu}{}_{\nu\rho} + L^{\mu}{}_{\nu\rho}\,. \tag{4}$$

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- Parts of the decomposition:
  - Levi-Civita connection of the metric:

$$\overset{\circ}{\Gamma}{}^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\right) \,. \tag{5}$$

  - Contortion:

$$K^{\mu}{}_{\nu\rho} = \frac{1}{2} \left(T_{\nu}{}^{\mu}{}_{\rho} + T_{\rho}{}^{\mu}{}_{\nu} - T^{\mu}{}_{\nu\rho}\right) \,. \tag{6}$$

  - Disformation:

$$L^{\mu}{}_{\nu\rho} = \frac{1}{2} \left(Q^{\mu}{}_{\nu\rho} - Q_{\nu}{}^{\mu}{}_{\rho} - Q_{\rho}{}^{\mu}{}_{\nu}\right) \,. \tag{7}$$

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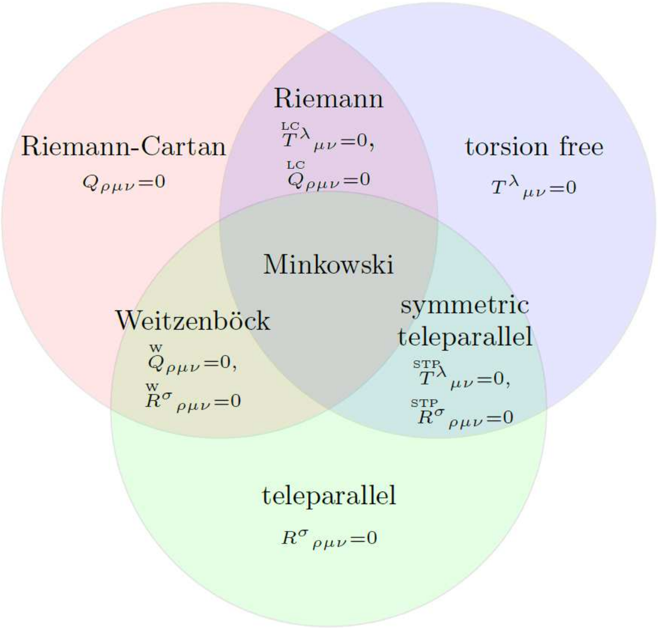
⇒ Define distortion:

$$M^{\mu}{}_{\nu\rho} = K^{\mu}{}_{\nu\rho} + L^{\mu}{}_{\nu\rho} \,. \tag{8}$$

The landscape of metric-affine of gravity



Cosmologically symmetric metric-affine geometry

1. Most general metric with cosmological symmetry:
   - Metric in space-time split:
   $$g_{\mu\nu} = -n_\mu n_\nu + h_{\mu\nu}\,. \tag{9}$$
   - Unit normal covector field:
   $$n_\mu \mathrm{d}x^\mu = -N\,\mathrm{d}t\,. \tag{10}$$
   - Spatial metric with curvature parameter $k \in \{-1, 0, 1\}$:
   $$h_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = A^2\left[\frac{\mathrm{d}r \otimes \mathrm{d}r}{1 - kr^2} + r^2(\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta \mathrm{d}\varphi \otimes \mathrm{d}\varphi)\right]\,. \tag{11}$$

⇒ Metric depends on lapse $N(t)$ and scale factor $A(t)$.

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   ⇒ Metric depends on lapse $N(t)$ and scale factor $A(t)$.
2. Most general affine connection with cosmological symmetry:
   - Connection characterized by cosmologically symmetric torsion and nonmetricity:
$$T^\mu{}_{\nu\rho} = \frac{2}{A}(\mathcal{T}_1 h^\mu_{[\nu} n_{\rho]} + \mathcal{T}_2 n_\sigma \varepsilon^{\sigma\mu}{}_{\nu\rho})\,, \quad Q_{\rho\mu\nu} = \frac{2}{A}(\mathcal{Q}_1 n_\rho n_\mu n_\nu + 2\mathcal{Q}_2 n_\rho h_{\mu\nu} + 2\mathcal{Q}_3 h_{\rho(\mu} n_{\nu)})\,. \tag{12}$$

   ⇒ Connection depends on five free functions $\mathcal{T}_1(t), \mathcal{T}_2(t), \mathcal{Q}_1(t), \mathcal{Q}_2(t), \mathcal{Q}_3(t)$.
   - Functions are further restricted by vanishing curvature, torsion, nonmetricity.

Structure of the teleparallel gravity action

- General structure of teleparallel gravity action with matter fields ξ:

$$S[g,\Gamma,\psi] = S_{\mathrm{g}}[g,\Gamma] + S_{\mathrm{L}}[g,\Gamma] + S_{\mathrm{m}}[g,\Gamma,\xi]\,, \tag{13}$$

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- Lagrange multiplier enforces constraints on the connection:
  - General teleparallel gravity:

$$S_{\mathrm{L}} = \int_M \tilde{r}_\mu{}^{\nu\rho\sigma} R^\mu{}_{\nu\rho\sigma} \mathrm{d}^4x\,, \tag{14}$$

  - Metric teleparallel gravity:

$$S_{\mathrm{L}} = \int_M (\tilde{r}_\mu{}^{\nu\rho\sigma} R^\mu{}_{\nu\rho\sigma} + \tilde{q}^{\mu\nu\rho} Q_{\mu\nu\rho}) \mathrm{d}^4x\,, \tag{15}$$

  - Symmetric teleparallel gravity:

$$S_{\mathrm{L}} = \int_M (\tilde{r}_\mu{}^{\nu\rho\sigma} R^\mu{}_{\nu\rho\sigma} + \tilde{t}_\mu{}^{\nu\rho} T^\mu{}_{\nu\rho}) \mathrm{d}^4x\,, \tag{16}$$

Energy-momentum-hypermomentum

- Variation of the matter part of the action:

$$\delta S_{\mathrm{m}} = \int_M \left(\frac{1}{2}\Theta^{\mu\nu}\delta g_{\mu\nu} + H_{\mu}{}^{\nu\rho}\delta\Gamma^{\mu}{}_{\nu\rho} + \Xi_I\delta\xi^I\right)\sqrt{-g}\mathrm{d}^4x\,, \tag{17}$$

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- Matter field equation: $\Xi_I = 0$.

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- Impose diffeomorphism invariance $\delta_X S_{\mathrm{m}} = 0$ and matter field equation:

$$\begin{aligned} 0 = \overset{\circ}{\nabla}_\nu\Theta_\mu{}^\nu &+ T^\sigma{}_{\mu\nu}(\nabla_\rho H_\sigma{}^{\nu\rho} - M^\tau{}_{\rho\tau}H_\sigma{}^{\nu\rho}) \\ &- \nabla_\nu(\nabla_\rho H_\mu{}^{\nu\rho} - M^\tau{}_{\rho\tau}H_\mu{}^{\nu\rho}) + M^\sigma{}_{\nu\sigma}(\nabla_\rho H_\mu{}^{\nu\rho} - M^\tau{}_{\rho\tau}H_\mu{}^{\nu\rho})\,. \end{aligned} \tag{18}$$

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- Energy-momentum-hypermomentum with cosmological symmetry:
  - Energy-momentum tensor with density ρ and pressure p:

$$\Theta_{\mu\nu} = \rho n_\mu n_\nu + p h_{\mu\nu}\,. \tag{19}$$

  - Hypermomentum with components $\phi, \chi, \psi, \omega, \zeta$:

$$H_{\rho\mu\nu} = \phi h_{\mu\rho}n_\nu + \chi h_{\nu\rho}n_\mu + \psi h_{\mu\nu}n_\rho + \omega n_\mu n_\nu n_\rho - \zeta n_\sigma \varepsilon^\sigma{}_{\mu\nu\rho}\,. \tag{20}$$

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- Barotropic equation of state: linear relation between variables.

General quadratic class of theories

- General quadratic gravitational action:

$$
\begin{aligned}
S_{\mathrm{g}} &= -\frac{1}{2\kappa^2}\int_M \Big[M^{\mu\nu\rho}(k_1 M_{\mu\nu\rho} + k_2 M_{\nu\rho\mu} + k_3 M_{\mu\rho\nu} + k_4 M_{\rho\nu\mu} + k_5 M_{\nu\mu\rho}) \\
&\quad + k_6 M_{\rho\mu}{}^{\mu} M^{\rho\nu}{}_{\nu} + k_7 M_{\mu\rho}{}^{\mu} M^{\nu\rho}{}_{\nu} + k_8 M^{\mu}{}_{\mu\rho} M_{\nu}{}^{\nu\rho} \\
&\quad + k_9 M_{\mu\rho}{}^{\mu} M_{\nu}{}^{\nu\rho} + k_{10} M^{\mu}{}_{\mu\rho} M^{\rho\nu}{}_{\nu} + k_{11} M_{\rho\mu}{}^{\mu} M^{\nu\rho}{}_{\nu}\Big]\sqrt{-g}\,\mathrm{d}^4x \\
&= -\frac{1}{2\kappa^2}\int_M \Big(a_1 T^{\mu\nu\rho} T_{\mu\nu\rho} + a_2 T^{\mu\nu\rho} T_{\rho\nu\mu} + a_3 T^{\mu}{}_{\mu\rho} T_{\nu}{}^{\nu\rho} \\
&\quad + c_1 Q^{\mu\nu\rho} Q_{\mu\nu\rho} + c_2 Q^{\mu\nu\rho} Q_{\rho\mu\nu} + c_3 Q^{\rho\mu}{}_{\mu} Q_{\rho\nu}{}^{\nu} + c_4 Q^{\mu}{}_{\mu\rho} Q_{\nu}{}^{\nu\rho} + c_5 Q^{\mu}{}_{\mu\rho} Q^{\rho\nu}{}_{\nu} \\
&\quad - b_1 Q^{\mu\nu\rho} T_{\rho\nu\mu} - b_2 Q^{\rho\mu}{}_{\mu} T^{\nu}{}_{\nu\rho} - b_3 Q_{\mu}{}^{\mu\rho} T^{\nu}{}_{\nu\rho}\Big)\sqrt{-g}\,\mathrm{d}^4x\,.
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⇒ Action depends on up to 11 parameters $k_{1\ldots11}$.

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- Parameters further restricted by consistency and phenomenology.

Cosmological dynamics as dynamical system

- Introduce helper variables to obtain first order ODE system.

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- Write matter variables $\rho, p, \ldots$ as quadratic in new quantities $D, P, \ldots$.

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⇒ General structure of cosmological field equations:

- Gravitational field equations:

$$A^a{}_b\dot{x}^b + B^a{}_{bc}x^b x^c = U^a{}_{IJ}y^I y^J\,. \tag{22}$$

- Energy-momentum-hypermomentum conservation:

$$V_I\dot{y}^I + W_{aI}x^a y^I = 0\,. \tag{23}$$

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⇒ Unified equation in terms of variable $\underline{z} = (x^a, y^I)$:

$$\dot{\underline{z}} = \underline{f}(\underline{z}, \underline{z}) \,. \tag{24}$$

Radial and angular dynamics

- Decomposition of variables into angular part (unit vector $\underline{n}$) and radial part (length Z):

$$\underline{z} = Z\underline{n}\,, \quad Z = \|\underline{z}\|\,, \quad \underline{n} = \frac{\underline{z}}{\|\underline{z}\|}\,. \tag{25}$$

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⇒ Dynamical equations:

- Radial equation:

$$\dot{Z} = Z^2\underline{f}(\underline{n},\underline{n})\cdot\underline{n}\,. \tag{26}$$

- Angular equation:

$$\dot{\underline{n}} = Z\left\{\underline{f}(\underline{n},\underline{n}) - [\underline{f}(\underline{n},\underline{n})\cdot\underline{n}]\,\underline{n}\right\}\,. \tag{27}$$

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⇒ Qualitative dynamics (up to positive factor Z) fully determined by $\underline{n}$.

Radial and angular dynamics

- Decomposition of variables into angular part (unit vector $\underline{n}$) and radial part (length Z):

$$\underline{z} = Z\underline{n}\,, \quad Z = \|\underline{z}\|\,, \quad \underline{n} = \frac{\underline{z}}{\|\underline{z}\|}\,. \tag{25}$$

⇒ Dynamical equations:

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- Unit sphere is compact: fixed points always exist.

Fixed points and projective fixed points

- Fixed points: $\dot{\underline{z}} = 0$.
  - $\Rightarrow$ $\underline{z} = 0$ is always a non-hyperbolic fixed point (saddle point).
  - $\Rightarrow$ If $\underline{z} = \underline{z}^*$ is a fixed point, then all $c\underline{z}^*$ with $c \in \mathbb{R}$ are fixed points.
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- Projective fixed points: $\dot{\underline{n}} = 0$.
  - $\Rightarrow$ Condition depends only on angular coordinates:

$$\frac{\dot{\underline{n}}}{Z} = \underline{f}(\underline{n}, \underline{n}) - [\underline{f}(\underline{n}, \underline{n}) \cdot \underline{n}]\, \underline{n} = 0\,. \tag{28}$$

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  - $\Rightarrow$ $\underline{n}^\star$ and $-\underline{n}^\star$ have opposite stability properties.
  - $\Rightarrow$ Since $\dot{\underline{n}} = 0$, $N^\star = \underline{f}(\underline{n}^\star, \underline{n}^\star) \cdot \underline{n}^\star$ is constant at a projective fixed point.
  - $\Rightarrow$ Radial dynamics $\dot{Z} = N^\star Z^2$ can be solved at projective fixed point:

$$Z(t) = \frac{1}{N^\star(t_0 - t)}\,. \tag{29}$$

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- Outlook:
  - Full classification of fixed points, stability, trajectories.
  - Study properties and dynamics of inflation and dark energy.
  - Study cosmological perturbations.
  - Generalization beyond quadratic teleparallel gravity theories.
  - Possible generalization to Bianchi spacetime models.