Selected Topics in the Theories of Gravity - Assignment 1

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1. **Energy-momentum tensor of the Klein-Gordon field**
   Consider the massive Klein-Gordon (scalar) field ϕ defined by the Lagrange function

$$L_M = -\frac{1}{2}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + m^2\phi^2\right)\,,$$

   where m is the mass of the field.

   (a) Derive the Klein-Gordon equation by variation of the action,

$$\frac{\delta S_M}{\delta\phi} = 0\,.$$

   (b) Calculate the energy-momentum tensor $T^{\mu\nu}$.

   (c) Show by explicit calculation that the energy-momentum tensor is covariantly conserved,

$$\nabla_\mu T^{\mu\nu} = 0\,.$$

2. **Lie derivative of the metric**
   Show by explicit calculation that the Lie derivative of the metric tensor satisfies

$$\mathcal{L}_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$$

   for a vector field ξ. You will need the formulas

$$\begin{aligned}\mathcal{L}_\xi \Phi^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s} = {}& \xi^\rho\partial_\rho\Phi^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s}\\ &- (\partial_\rho\xi^{\mu_1})\Phi^{\rho\cdots\mu_r}{}_{\nu_1\ldots\nu_s} - \ldots - (\partial_\rho\xi^{\mu_r})\Phi^{\mu_1\cdots\rho}{}_{\nu_1\ldots\nu_s}\\ &+ (\partial_{\nu_1}\xi^\rho)\Phi^{\mu_1\cdots\mu_r}{}_{\rho\ldots\nu_s} + \ldots + (\partial_{\nu_s}\xi^\rho)\Phi^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\rho}\end{aligned}$$

   and

$$\begin{aligned}\nabla_\rho \Phi^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s} = {}& \partial_\rho\Phi^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s}\\ &+ \Gamma^{\mu_1}{}_{\rho\sigma}\Phi^{\sigma\cdots\mu_r}{}_{\nu_1\ldots\nu_s} + \ldots + \Gamma^{\mu_r}{}_{\rho\sigma}\Phi^{\mu_1\cdots\sigma}{}_{\nu_1\ldots\nu_s}\\ &- \Gamma^{\sigma}{}_{\rho\nu_1}\Phi^{\mu_1\cdots\mu_r}{}_{\sigma\ldots\nu_s} - \ldots - \Gamma^{\sigma}{}_{\rho\nu_s}\Phi^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\sigma}\end{aligned}$$

   with

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$$

   for tensor fields $\Phi^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s}$.