Selected Topics in the Theories of Gravity - Assignment 2

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1. **Energy conditions**

   (a) Prove that the strong energy condition,

   $$\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) \xi^\mu \xi^\nu \geq 0 \tag{1}$$

   for all timelike vectors ξ^μ, can be expressed in terms of the eigenvalues of the energy-momentum tensor as

   $$\rho + \sum_{i=1}^{3} p_i \geq 0 \quad \text{and} \quad \rho + p_i \geq 0 \, \forall i = 1, 2, 3 \,. \tag{2}$$

   (b) Prove these statements using the eigenvalues of the energy-momentum tensor:
   - If the dominant energy condition holds, also the weak energy condition holds.
   - If the weak energy condition holds, also the null energy condition holds.
   - If the strong energy condition holds, also the null energy condition holds.

   (c) Show by example that the weak energy condition does not follow from the strong energy condition.

2. **Schwarzschild wormhole**
   Consider the Schwarzschild wormhole, i.e. the wormhole defined by the metric

   $$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2 [d\theta^2 + \sin^2\theta \, d\phi^2] \,, \tag{3}$$

   where $b(r) = r_0 = \text{const.}$ and $e^{2\Phi(r)} = 1 - r_0/r$. As in the lecture, the coordinate r has the range $r \geq r_0$ in two copies of flat Minkowski spacetime which are glued together at the throat $r = r_0$.

   (a) Show that the rest energy density ρ, the radial pressure p_r and the lateral pressure p_l (and thus the energy-momentum tensor $T_{\mu\nu}$) vanish. Does this solution satisfy the energy conditions?

   (b) Does this solution satisfy the conditions on a traversable wormhole? Explain your answer.