Selected Topics in the Theories of Gravity - Assignment 3

Manuel Hohmann

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1. **Tensor densities**

   (a) Show that the product of two tensor densities $\mathfrak{A}, \mathfrak{B}$ of weights w and w' is a tensor density of weight $w + w'$.

   (b) Show that the Leibnitz rule $\nabla_\mu(\mathfrak{A}\mathfrak{B}) = (\nabla_\mu\mathfrak{A})\mathfrak{B} + \mathfrak{A}(\nabla_\mu\mathfrak{B})$ holds also for tensor densities.

2. **Metric tensor density**

   (a) Use your knowledge about variation of the metric tensor to show that

   $$\delta\sqrt{-g} = \frac{1}{2}g_{\mu\nu}\delta\mathfrak{g}^{\mu\nu}\,. \tag{1}$$

   (b) Calculate $\nabla_\mu\sqrt{-g}$ and $\nabla_\mu\mathfrak{g}^{\rho\sigma}$ in terms of general (not necessarily metric-compatible) connection coefficients $\Gamma^\mu{}_{\nu\rho}$.

   (c) Use equation (1) in the form

   $$\partial_\mu\sqrt{-g} = \frac{1}{2}g_{\rho\sigma}\partial_\mu\mathfrak{g}^{\rho\sigma} \tag{2}$$

   and the result $\nabla_\rho\mathfrak{g}^{\mu\nu} = 0$ from the lecture to show that $\nabla_\mu\sqrt{-g} = 0$.

   (d) Use

   $$\nabla_\rho\left(\sqrt{-g}\delta^\mu_\nu\right) = \nabla_\rho\left(g_{\nu\sigma}\mathfrak{g}^{\sigma\mu}\right) \tag{3}$$

   to show that $\nabla_\rho g_{\mu\nu} = 0$.