Selected Topics in the Theories of Gravity - Assignment 4

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27. February 2014

1. **Post-Newtonian energy-momentum conservation**
   Consider the perfect fluid energy-momentum tensor
   $$T^{\mu\nu} = (\rho + \rho\Pi + p)u^\mu u^\nu + pg^{\mu\nu}\,, \tag{1}$$
   where $\rho \sim \Pi \sim \mathcal{O}(2)$ and $p \sim \mathcal{O}(4)$.
   (a) Use the approximation
   $$g_{\mu\nu} = \eta_{\mu\nu} + h^{(2)}_{\mu\nu} \tag{2}$$
   to express $g^{\mu\nu}$ in terms of $\eta^{\mu\nu}$ and $h^{(2)}_{\mu\nu}$. Keep only terms up to linear order in $h^{(2)}_{\mu\nu}$.
   (b) Calculate $\Gamma^i{}_{00}$ up to second order $\mathcal{O}(2)$.
   (c) Use $u^i = u^0 v^i$ and the normalization
   $$u^\mu u^\nu g_{\mu\nu} = -1 \tag{3}$$
   and write u^0 and u^i in terms of v^i and $h^{(2)}_{00} = 2U$. Keep only terms up to second velocity order $\mathcal{O}(2)$.
   (d) Calculate the components T^{00}, T^{0i}, T^{ij} up to the fourth velocity order $\mathcal{O}(4)$.
   (e) Calculate the conservation equations
   $$\nabla_\mu T^{\mu 0} = 0\,, \quad \nabla_\mu T^{\mu i} = 0 \tag{4}$$
   and keep only the lowest order terms in each equation. Use the fact that $\nabla_\mu = \partial_\mu + \mathcal{O}(2)$ to determine which Christoffel symbols you need to take into account. Show that these reproduce the conservation equations
   $$0 = \nabla_\mu T^{\mu 0} = \rho_{,0} + (\rho v_i)_{,i} + \mathcal{O}(5)\,, \tag{5a}$$
   $$0 = \nabla_\mu T^{\mu i} = \rho\frac{dv_i}{dt} + p_{,i} - \rho U_{,i} + \mathcal{O}(6)\,. \tag{5b}$$

2. **Post-Newtonian potentials**
   Use the continuity equation (5a) to show that the potentials
   $$U(t,\vec{x}) = \int d^3x'\,\frac{\rho(t,\vec{x}')}{|\vec{x}-\vec{x}'|} \sim \mathcal{O}(2)\,, \tag{6a}$$
   $$V_i(t,\vec{x}) = \int d^3x'\,\frac{\rho(t,\vec{x}')v_i(t,\vec{x}')}{|\vec{x}-\vec{x}'|} \sim \mathcal{O}(3)\,, \tag{6b}$$
   $$W_i(t,\vec{x}) = \int d^3x'\,\frac{\rho(t,\vec{x}')v_j(t,\vec{x}')(x_i - x'_i)(x_j - x'_j)}{|\vec{x}-\vec{x}'|^3} \sim \mathcal{O}(3) \tag{6c}$$
   satisfy the relations
   $$U_{,0} = -V_{i,i} = W_{i,i}\,. \tag{7}$$