Selected Topics in the Theories of Gravity - Assignment 7

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1. **Plane wave**
   Consider the plane wave for the tensor field χ given by
   $$\chi(t,x,y,z) = \hat{\chi}e^{i\omega(t-z)} = \hat{\chi}e^{i\omega u} \tag{1}$$
   with retarded time $u = t - z$, frequency ω and complex amplitude $\hat{\chi}$.
   (a) Show that
   $$\chi_{,\mu} = \partial_\mu\chi = -l_\mu\dot{\chi}\,, \tag{2}$$
   where l^μ is the first basis vector in the Newman-Penrose basis given by
   $$l^\mu = (1,0,0,1)\,, \quad n^\mu = \frac{1}{2}(1,0,0,-1)\,, \quad m^\mu = \frac{1}{\sqrt{2}}(0,1,i,0)\,, \quad \bar{m}^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0) \tag{3}$$
   in the (t,x,y,z) basis and the dot denotes the derivative with respect to u.
   (b) Calculate the quantities
   $$\chi_{,l} = l^\mu\chi_{,\mu}\,, \quad \chi_{,n} = n^\mu\chi_{,\mu}\,, \quad \chi_{,m} = m^\mu\chi_{,\mu}\,, \quad \chi_{,\bar{m}} = \bar{m}^\mu\chi_{,\mu}\,. \tag{4}$$

2. **Riemann tensor**
   (a) Calculate the Riemann tensor of a nearly flat metric
   $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{5}$$
   up to linear order in the metric perturbations $h_{\mu\nu}$.
   (b) Rewrite the linearized Riemann tensor in the Newman-Penrose basis and assume that the metric perturbations are given by a plane wave
   $$h_{\mu\nu}(t,x,y,z) = \hat{h}_{\mu\nu}e^{i\omega(t-z)} = \hat{h}_{\mu\nu}e^{i\omega u} \tag{6}$$
   with retarded time $u = t - z$ for a single Fourier mode with frequency ω and complex amplitude $\hat{h}_{\mu\nu}$. Show that the only non-vanishing components (up to symmetry) of the Riemann tensor are
   $$\Psi_2 = -\frac{1}{6}R_{nlnl}\,, \quad \Psi_3 = -\frac{1}{2}R_{nln\bar{m}} = -\frac{1}{2}\overline{R_{nlnm}}\,, \tag{7}$$
   $$\Psi_4 = -R_{n\bar{m}n\bar{m}} = -\overline{R_{nmnm}}\,, \quad \Phi_{22} = -R_{nmn\bar{m}} \tag{8}$$
   and express them in terms of the metric components $h_{nn}, h_{nl}, \ldots$.
   (c) Calculate the linearized Ricci tensor $R_{\mu\nu}$ in the Newman-Penrose basis and show that from the vacuum field equations $R_{\mu\nu} = 0$ of general relativity follows $\Psi_2 = \Psi_3 = \Phi_{22} = 0$. What is the E(2) class of general relativity?