Selected Topics in the Theories of Gravity

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1 Einstein-Hilbert action

The dynamics of general relativity can be derived from an action of the form

$$S[g, \Phi] = S_G[g] + S_M[g, \Phi]\,, \tag{1.1}$$

where g denotes the Lorentzian metric of spacetime and Φ denotes some matter fields. The terms here are the Einstein-Hilbert action

$$S_G[g] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R - 2\Lambda)\,, \tag{1.2}$$

where R is the Ricci scalar and Λ is the cosmological constant, and some matter action

$$S_M[g, \Phi] = \int d^4x\sqrt{-g}\,L_M[g, \Phi] \tag{1.3}$$

with Lagrange function $L_M[g, \Phi]$. By variation of the total action with respect to the metric $g_{\mu\nu}$ we can derive the Einstein equations. We will start with the gravitational part $S_G[g]$, which we write in the form

$$S_G[g] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(g^{\mu\nu}R_{\mu\nu} - 2\Lambda)\,, \tag{1.4}$$

where $R_{\mu\nu}$ is the Ricci tensor. The variation of the action then takes the form

$$\delta S_G = \frac{1}{16\pi G}\int d^4x\left[\delta\sqrt{-g}\,(g^{\mu\nu}R_{\mu\nu} - 2\Lambda) + \sqrt{-g}\,(\delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu})\right]\,. \tag{1.5}$$

We now see that we need to calculate the variation of three terms. We start with the determinant term coming from the volume form. Its variation is given by

$$\delta\sqrt{-g} = \frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu}\,. \tag{1.6}$$

The variation of the inverse metric takes is given by

$$\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}\,. \tag{1.7}$$

We finally need to calculate the variation of the Ricci tensor. Here we will use a trick to simplify the calculation. Recall that the Ricci tensor is a contraction of the Riemann tensor,

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}\,, \tag{1.8}$$

and that the Riemann tensor is given by

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\rho\tau}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\sigma\tau}\Gamma^{\tau}{}_{\nu\rho} \,, \tag{1.9}$$

where $\Gamma^{\mu}{}_{\nu\rho}$ are the Christoffel symbols. The variation of the Riemann tensor is thus given by

$$\delta R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \delta\Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \delta\Gamma^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\rho\tau}\Gamma^{\tau}{}_{\nu\sigma} + \Gamma^{\mu}{}_{\rho\tau}\delta\Gamma^{\tau}{}_{\nu\sigma} - \delta\Gamma^{\mu}{}_{\sigma\tau}\Gamma^{\tau}{}_{\nu\rho} - \Gamma^{\mu}{}_{\sigma\tau}\delta\Gamma^{\tau}{}_{\nu\rho} \,. \tag{1.10}$$

We now use the fact that $\delta\Gamma^{\mu}{}_{\nu\rho}$ is the difference between to connections (the Levi-Civita connections of $g_{\mu\nu}$ and $g_{\mu\nu} + \delta g_{\mu\nu}$), and hence is a tensor. Its covariant derivative is given by

$$\nabla_\rho \delta\Gamma^{\mu}{}_{\nu\sigma} = \partial_\rho \delta\Gamma^{\mu}{}_{\nu\sigma} + \Gamma^{\mu}{}_{\rho\tau}\delta\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\tau}{}_{\rho\nu}\delta\Gamma^{\mu}{}_{\tau\sigma} - \Gamma^{\tau}{}_{\rho\sigma}\delta\Gamma^{\mu}{}_{\nu\tau} \,. \tag{1.11}$$

One can now easily check that

$$\delta R^{\mu}{}_{\nu\rho\sigma} = \nabla_\rho \delta\Gamma^{\mu}{}_{\nu\sigma} - \nabla_\sigma \delta\Gamma^{\mu}{}_{\nu\rho} \,. \tag{1.12}$$

For the variation of the Ricci tensor we thus find

$$\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^{\rho}{}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\rho}{}_{\mu\rho} \,. \tag{1.13}$$

Contracted with the inverse metric this yields

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\left(\nabla_\rho \delta\Gamma^{\rho}{}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\rho}{}_{\mu\rho}\right) = \nabla_\rho\left(g^{\mu\nu}\delta\Gamma^{\rho}{}_{\mu\nu} - g^{\mu\rho}\delta\Gamma^{\nu}{}_{\mu\nu}\right) \,, \tag{1.14}$$

where we used the fact that the metric, and thus also its inverse, is covariantly constant, $\nabla_\rho g^{\mu\nu} = 0$. The expression we obtain is a total covariant divergence. Its integral

$$\int d^4x \sqrt{-g}\, \nabla_\rho\left(g^{\mu\nu}\delta\Gamma^{\rho}{}_{\mu\nu} - g^{\mu\rho}\delta\Gamma^{\nu}{}_{\mu\nu}\right) \tag{1.15}$$

therefore vanishes. In summary we thus find the variation

$$\delta S_G = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}(R - 2\Lambda) - R^{\mu\nu}\right]\delta g_{\mu\nu} \,. \tag{1.16}$$

In a similar way we can calculate the variation of the matter action (1.3),

$$\delta S_M = \int d^4x \frac{\delta}{\delta g_{\mu\nu}}\left[\sqrt{-g}\, L_M\right]\delta g_{\mu\nu} = \frac{1}{2}\int d^4x \sqrt{-g}\, T^{\mu\nu}\delta g_{\mu\nu} \,, \tag{1.17}$$

where we have introduced the energy-momentum tensor

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\, L_M)}{\delta g_{\mu\nu}} \,. \tag{1.18}$$

From the condition that δS vanishes for arbitrary variations $\delta g_{\mu\nu}$ we then find, after lowering the indices with the metric,

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \,. \tag{1.19}$$

These are the Einstein equations.

2 Covariant energy-momentum conservation

It is well known that the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{2.1}$$

satisfies the Bianchi identity

$$\nabla_\mu G^{\mu\nu} = 0\,. \tag{2.2}$$

Further, the metric is covariantly constant, $\nabla_\rho g^{\mu\nu} = 0$. From the Einstein equations (1.19) thus follows that also the energy-momentum tensor $T_{\mu\nu}$ must be covariantly conserved,

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{2.3}$$

There is an elegant and more fundamental way to see this, which follows directly from the definition (1.18). In order for the matter action (1.3) to have a physical meaning which is independent of the choice of coordinates, it must be invariant under an infinitesimal coordinate change of the form

$$x^\mu \mapsto x^\mu + \xi^\mu\,, \tag{2.4}$$

where ξ is a vector field. This coordinate transformation, or diffeomorphism, induces a "shift" of tensor fields Φ by the amount

$$\begin{aligned} \delta_\xi \Phi^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} &= \xi^\rho \partial_\rho \Phi^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} \\ &\quad - (\partial_\rho \xi^{\mu_1}) \Phi^{\rho \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} - \ldots - (\partial_\rho \xi^{\mu_r}) \Phi^{\mu_1 \ldots \rho}{}_{\nu_1 \ldots \nu_s} \\ &\quad + (\partial_{\nu_1} \xi^\rho) \Phi^{\mu_1 \ldots \mu_r}{}_{\rho \ldots \nu_s} + \ldots + (\partial_{\nu_s} \xi^\rho) \Phi^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \rho} \\ &= \mathcal{L}_\xi \Phi^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s}\,, \end{aligned} \tag{2.5}$$

which is called the Lie derivative. We can now calculate the change of the matter action (1.3) under this diffeomorphism. It takes the form

$$\delta_\xi S = \int d^4x \left[\frac{\delta(\sqrt{-g}\, L_M)}{\delta g_{\mu\nu}} \delta_\xi g_{\mu\nu} + \frac{\delta(\sqrt{-g}\, L_M)}{\delta \Phi} \delta_\xi \Phi \right]. \tag{2.6}$$

We first take a look at the second term. Since $\sqrt{-g}$ does not depend on Φ, we find

$$\frac{\delta(\sqrt{-g}\, L_M)}{\delta \Phi} = \sqrt{-g} \frac{\delta L_M}{\delta \Phi}\,, \tag{2.7}$$

which vanishes because of the matter field equations

$$\frac{\delta L_M}{\delta \Phi} = 0\,. \tag{2.8}$$

This leaves us with only the first term. Here we need to calculate the Lie derivative

$$\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 2\nabla_{(\mu} \xi_{\nu)}\,. \tag{2.9}$$

Further, recall that by definition (1.18) of the energy-momentum tensor

$$\frac{\delta(\sqrt{-g}\, L_M)}{\delta g_{\mu\nu}} = \frac{\sqrt{-g}}{2} T^{\mu\nu}\,. \tag{2.10}$$

From this we find

$$\begin{aligned}
\delta_\xi S &= \int d^4x \frac{\delta(\sqrt{-g}\,L_M)}{\delta g_{\mu\nu}} \delta_\xi g_{\mu\nu} \\
&= \int d^4x \sqrt{-g}\, T^{\mu\nu} \nabla_\mu \xi_\nu \\
&= -\int d^4x \sqrt{-g}\, (\nabla_\mu T^{\mu\nu}) \xi_\nu \,,
\end{aligned} \tag{2.11}$$

where the last line follows from integration by parts. This vanishes for arbitrary vector fields ξ if and only if $\nabla_\mu T^{\mu\nu}$ vanishes identically, i.e., if the covariant energy-momentum conservation (2.3) holds.

3 Example: Maxwell field

As an example for a matter field we consider the Maxwell field without charges given by the Lagrange function

$$L_M = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{16\pi} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \,. \tag{3.1}$$

The calculation of the energy-momentum tensor is straightforward:

$$\begin{aligned}
T^{\mu\nu} &= \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\,L_M)}{\delta g_{\mu\nu}} \\
&= -\frac{1}{8\pi\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left[\sqrt{-g} F_{\rho\sigma} F_{\pi\tau} g^{\rho\pi} g^{\sigma\tau}\right] \\
&= -\frac{1}{8\pi\sqrt{-g}} \left[\frac{\delta\sqrt{-g}}{\delta g_{\mu\nu}} g^{\rho\pi} g^{\sigma\tau} + \sqrt{-g}\left(\frac{\delta g^{\rho\pi}}{\delta g_{\mu\nu}} g^{\sigma\tau} + g^{\rho\pi} \frac{\delta g^{\sigma\tau}}{\delta g_{\mu\nu}}\right)\right] F_{\rho\sigma} F_{\pi\tau} \\
&= -\frac{1}{8\pi} \left[\frac{1}{2} g^{\mu\nu} g^{\rho\pi} g^{\sigma\tau} - g^{\mu\rho} g^{\nu\pi} g^{\sigma\tau} - g^{\rho\pi} g^{\mu\sigma} g^{\nu\tau}\right] F_{\rho\sigma} F_{\pi\tau} \\
&= \frac{1}{4\pi} \left(F^{\mu\rho} F^{\nu\sigma} g_{\rho\sigma} - \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right) \,.
\end{aligned} \tag{3.2}$$

Lowering the indices yields

$$T_{\mu\nu} = \frac{1}{4\pi} \left(F_{\mu\rho} F_{\nu\sigma} g^{\rho\sigma} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right) \,. \tag{3.3}$$

We finally show by explicit calculation that $T_{\mu\nu}$ is covariantly conserved. Also this calculation is straightforward:

$$\begin{aligned}
4\pi \nabla_\mu T^{\mu\nu} &= \nabla_\mu \left(F^{\mu\rho} F^{\nu\sigma} g_{\rho\sigma} - \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right) \\
&= \nabla_\mu F^{\mu\rho} F^{\nu\sigma} g_{\rho\sigma} + F^{\mu\rho} \nabla_\mu F^{\nu\sigma} g_{\rho\sigma} - \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} g^{\pi\tau} (\nabla_\mu F_{\rho\pi} F_{\sigma\tau} + F_{\rho\pi} \nabla_\mu F_{\sigma\tau}) \,.
\end{aligned} \tag{3.4}$$

Here $\nabla_\mu F^{\mu\rho}$ vanishes because of the Gauss-Ampére law

$$\nabla_\mu F^{\mu\nu} = 0 \,. \tag{3.5}$$

Further, the last two terms can be combined because of the symmetry of $g^{\rho\sigma}$ and $g^{\pi\tau}$. From this we get

$$4\pi\nabla_\mu T^{\mu\nu} = g^{\mu\pi}g^{\nu\tau}g^{\rho\sigma}F_{\pi\rho}\nabla_\mu F_{\tau\sigma} - \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}g^{\pi\tau}F_{\rho\pi}\nabla_\mu F_{\sigma\tau}\,. \tag{3.6}$$

In the first term we now exchange the indices μ and τ by renaming. In the second term we use the antisymmetry of $F_{\rho\pi}$. This yields

$$4\pi\nabla_\mu T^{\mu\nu} = g^{\mu\nu}g^{\pi\tau}g^{\rho\sigma}F_{\pi\rho}\nabla_\tau F_{\mu\sigma} + \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}g^{\pi\tau}F_{\pi\rho}\nabla_\mu F_{\sigma\tau}\,. \tag{3.7}$$

This can be rearranged to give

$$4\pi\nabla_\mu T^{\mu\nu} = \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}g^{\pi\tau}F_{\pi\rho}\left(\nabla_\tau F_{\mu\sigma} + \nabla_\tau F_{\mu\sigma} + \nabla_\mu F_{\sigma\tau}\right)\,, \tag{3.8}$$

where we wrote the term $\nabla_\tau F_{\mu\sigma}$ twice to compensate for the factor $1/2$ in front of the whole expression. With one of these two identical terms we then make two transformations. First, we use the antisymmetry of $F_{\mu\sigma}$ to exchange the indices μ and σ, which gives us a factor -1. We then exchange the indices τ and σ, which gives us another factor -1, because they are contracted with the antisymmetric $F_{\pi\rho}$ via $g^{\rho\sigma}$ and $g^{\pi\tau}$. Both factors cancel, so that this finally yields

$$4\pi\nabla_\mu T^{\mu\nu} = \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}g^{\pi\tau}F_{\pi\rho}\left(\nabla_\sigma F_{\tau\mu} + \nabla_\tau F_{\mu\sigma} + \nabla_\mu F_{\sigma\tau}\right)\,. \tag{3.9}$$

Now the term in brackets vanishes because of the Gauss-Faraday law

$$\nabla_{[\rho}F_{\mu\nu]} = 0\,. \tag{3.10}$$

We thus see that the energy-momentum tensor of the Maxwell field is indeed covariantly conserved.