Selected Topics in the Theories of Gravity

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1 Energy conditions

There are four different, but related energy conditions which can be imposed on the energy-momentum tensor [1]. Matter satisfying these conditions is denoted *ordinary* matter, while matter violating these conditions is called *exotic* matter.

1.1 Weak energy condition (WEC)

The weak energy condition states that for any observer the measured energy density is non-negative. Since the energy density measured by an observer with four-velocity ξ^μ is given by $T_{\mu\nu}\xi^\mu\xi^\nu$, this means that

$$T_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \tag{1.1}$$

for all future timelike vectors ξ^μ.

1.2 Dominant energy condition (DEC)

The dominant energy condition states that in addition to the weak energy condition holding true, the energy flow that an observer measures is always future-directed and does not exceed the speed of light. For an observer with four-velocity ξ^μ the energy flow is given by its four-current density

$$j^\mu = -T^\mu{}_\nu\xi^\nu\,. \tag{1.2}$$

The dominant energy condition thus requires that for any future timelike vector ξ^μ, the current density must be a future causal (i.e., timelike or lightlike) vector.

1.3 Strong energy condition (SEC)

The strong energy condition has a similar form as the weak energy condition. It requires that

$$\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)\xi^\mu\xi^\nu \geq 0 \tag{1.3}$$

for all future timelike vectors ξ^μ. Via the Einstein equations this expression is related to the trace of the tidal tensor $E[\xi]_{\mu\nu}$,

$$g^{\mu\nu}E[\xi]_{\mu\nu} = g^{\mu\nu}R_{\mu\rho\nu\sigma}\xi^\rho\xi^\sigma = R_{\rho\sigma}\xi^\rho\xi^\sigma\,. \tag{1.4}$$

It basically states that any observer measures gravity as an attractive force.

1.4 Null energy condition (NEC)

The null energy condition is a limiting case of the weak (or strong) energy condition. It states that (1.1) (or equivalently (1.3)) holds for any null vector ξ^μ, i.e., $g_{\mu\nu}\xi^\mu\xi^\nu = 0$.

1.5 Relation of the energy conditions

The energy conditions given above are not independent of each other. Indeed one can show that:

- If the dominant energy condition holds, also the weak energy condition holds.
- If the weak energy condition holds, also the null energy condition holds.
- If the strong energy condition holds, also the null energy condition holds.

Note that the weak energy condition does *not* follow from the strong energy condition. These relations can be summarized in the following diagram:

$$\begin{tikzcd}
\text{DEC} \arrow[d, Rightarrow] & & \text{SEC} \arrow[ddl, Rightarrow] \\
\text{WEC} \arrow[dr, Rightarrow] & & \\
& \text{NEC} &
\end{tikzcd}$$

2 Eigenvalues of the energy-momentum tensor

Using the definitions given above one can easily see that an energy condition is violated when there exists a vector ξ^μ which violates the corresponding inequality. However, it is more difficult to show that an energy condition is satisfied, because one must prove that no such vector exists. A helpful method is thus to reformulate the energy conditions in terms of the eigenvalues of the energy-momentum tensor.
An eigenvector ξ^μ of the energy-momentum tensor with eigenvalue λ is a vector which satisfies

$$T^\mu{}_\nu \xi^\nu = \lambda \xi^\mu \,. \tag{2.1}$$

The eigenvalues of the energy-momentum tensor are those λ for which a non-trivial eigenvector $\xi^\mu \neq 0$ exists. Note that this is only the case when

$$\det(T^\mu{}_\nu - \lambda\delta^\mu_\nu) = 0 \,, \tag{2.2}$$

or equivalently,

$$\det(T_{\mu\nu} - \lambda g_{\mu\nu}) = 0 \,. \tag{2.3}$$

Since this eigenvalue equation is a fourth order algebraic equation, it has four not necessarily different, possibly complex solutions. The complete classification of eigenvalues and eigenvectors can be tedious [2]; in the following discussion we restrict ourselves to the special case that all eigenvalues are real, and there exists one timelike eigenvector ${\xi_0}^\mu$ and three spacelike eigenvectors ${\xi_i}^\mu$.

We can choose a set of eigenvectors which makes calculations a bit simpler. Recall from linear algebra that eigenvectors belonging to different eigenvalues are always orthogonal to each other, since $T_{\mu\nu}$ is symmetric,

$$(\lambda - \lambda')g_{\mu\nu}\xi^\mu\xi'^\nu = g_{\mu\nu}\left(T^\mu{}_\sigma\xi^\sigma\xi'^\nu - T^\nu{}_\sigma\xi^\mu\xi'^\sigma\right) = (T_{\nu\mu} - T_{\mu\nu})\xi^\mu\xi'^\nu = 0\,. \tag{2.4}$$

Further, if two (or more) eigenvalues are the same, we can always choose a set of orthogonal eigenvectors for these eigenvalues. Finally, we can normalize these eigenvectors using the metric. In summary, we can always choose the vectors $\xi_\alpha{}^\mu$ such that they satisfy

$$g_{\mu\nu}\xi_\alpha{}^\mu\xi_\beta{}^\nu = \eta_{\alpha\beta}\,, \tag{2.5}$$

i.e., they form an orthonormal basis. Using this basis the energy-momentum tensor always takes the form

$$T_{\mu\nu} = \rho\xi_{0\mu}\xi_{0\nu} + p_1\xi_{1\mu}\xi_{1\nu} + p_2\xi_{2\mu}\xi_{2\nu} + p_3\xi_{3\mu}\xi_{3\nu}\,, \tag{2.6}$$

where ρ is called the rest energy and p_i are called the principal pressures. We will now formulate the energy conditions in terms of these eigenvalues.

We first take a look at the weak energy condition. Any timelike vector ξ^μ can be written in the basis $\xi_\alpha{}^\mu$ as

$$\xi^\mu = v^\alpha\xi_\alpha{}^\mu\,, \tag{2.7}$$

where the coefficients v^α satisfy

$$\eta_{\alpha\beta}v^\alpha v^\beta = -v^2 < 0\,. \tag{2.8}$$

For the inequality (1.1) we then find

$$\begin{aligned} T_{\mu\nu}\xi^\mu\xi^\nu &= T_{\mu\nu}\xi_\alpha{}^\mu\xi_\beta{}^\mu v^\alpha v^\beta \\ &= \rho(v^0)^2 + p_1(v^1)^2 + p_2(v^2)^2 + p_3(v^3)^2 \\ &= \rho\left(v^2 + (v^1)^2 + (v^2)^2 + (v^3)^2\right) + p_1(v^1)^2 + p_2(v^2)^2 + p_3(v^3)^2 \\ &= \rho v^2 + (\rho + p_1)(v^1)^2 + (\rho + p_2)(v^2)^2 + (\rho + p_3)(v^3)^2 \end{aligned} \tag{2.9}$$

Here $v^2, (v^1)^2, (v^2)^2, (v^3)^2$ are arbitrary positive numbers. The result is thus non-negative for all timelike vectors if and only if

$$\rho \geq 0 \quad \text{and} \quad \rho + p_i \geq 0\,\forall i = 1, 2, 3\,. \tag{2.10}$$

This is the weak energy condition in terms of the rest energy density and principal pressures.

One can proceed similarly for the other energy conditions. One finds that the strong energy condition takes the form

$$\rho + \sum_{i=1}^{3} p_i \geq 0 \quad \text{and} \quad \rho + p_i \geq 0\,\forall i = 1, 2, 3\,, \tag{2.11}$$

the dominant energy condition takes the form

$$\rho \geq 0 \quad \text{and} \quad -\rho \leq p_i \leq \rho\,\forall i = 1, 2, 3\,, \tag{2.12}$$

and the null energy condition takes the form

$$\rho + p_i \geq 0\,\forall i = 1, 2, 3\,. \tag{2.13}$$

3 Wormholes

Consider the static, spherically symmetric spacetime metric in spherical coordinates (t, l, θ, ϕ) given by

$$ds^2 = -e^{2\Phi(l)}dt^2 + dl^2 + r^2(l)[d\theta^2 + \sin^2\theta\, d\phi^2]\,, \tag{3.1}$$

where the coordinate l ranges from $-\infty$ to $+\infty$ and Φ, r are free functions of l. We use this metric to model a traversable wormhole which connects two copies of asymptotically flat Minkowski spacetime. The wormhole throat should be located at $l = 0$, and the two asymptotically flat spacetimes should be the regions $l > 0$ and $l < 0$. We get a few conditions on Φ and r:

- Since $r(l)$ measures the radius of a sphere with constant distance l from the throat, it must always be positive.
- At the throat $l = 0$ the radius must have a positive minimal value, i.e., $r(0) = r_0 > 0$, $r'(0) = 0$ and $r''(0) > 0$.
- Spacetime must be asymptotically flat on both sides of the wormhole,

$$\lim_{l\to\pm\infty} \frac{r(l)}{|l|} = 1 \quad \text{and} \quad \lim_{l\to\pm\infty} \Phi(l) = 1\,. \tag{3.2}$$

- The wormhole must not have any horizons, i.e., $\Phi(l)$ must be everywhere finite.

These coordinates are quite intuitive for defining a wormhole. However, they turn out to be less practical for calculations. We therefore choose Schwarzschild-like coordinates on the two patches $l \geq 0$ and $l \leq 0$. We substitute the distance coordinate l by the radial coordinate $r \geq r_0$ on both sides of the wormhole. On the two patches the metric then takes the form

$$ds^2 = -e^{2\Phi_\pm(r)}dt^2 + \frac{dr^2}{1 - b_\pm(r)/r} + r^2[d\theta^2 + \sin^2\theta\, d\phi^2]\,, \tag{3.3}$$

where we now have free functions $\Phi_\pm$ and $b_\pm$, called the lapse and shape functions, on both wormhole sides. For simplicity we assume that the wormhole is symmetric and drop the subscripts $\pm$. The relation between the functions $r(l)$ and $b(r)$ is given by the proper distance,

$$dl^2 = \frac{dr^2}{1 - b(r)/r}\,, \tag{3.4}$$

so that we have

$$\frac{dr}{dl} = \sqrt{1 - \frac{b(r)}{r}}\,, \quad \frac{d^2r}{dl^2} = \frac{b(r) - b'(r)r}{2r^2}\,. \tag{3.5}$$

In these coordinates the conditions thus take the form:

- At the throat $r = r_0$ we have $b(r_0) = r_0$ and $b(r_0)/r_0 > b'(r_0)$.
- Spacetime must be asymptotically flat on both sides of the wormhole,

$$\lim_{r\to\infty} \frac{b(r)}{r} = 0 \quad \text{and} \quad \lim_{r\to\infty} \Phi(r) = 1\,. \tag{3.6}$$

- The wormhole must not have any horizons, i.e., $\Phi(r)$ must be everywhere finite.

One can now take the metric and calculate the Einstein tensor $G_{\mu\nu}$. This is a simple, but tedious calculation [3, 4, 5]. The Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ then yield the energy-momentum tensor.
After the calculation it turns out that the energy-momentum tensor is in fact diagonal in the coordinate basis of the coordinates (t, r, θ, ϕ) we use here; this also holds true for the metric. We can thus choose an orthonormal basis,

$$\xi_{\hat{t}} = e^{-\Phi}\partial_t \,, \quad \xi_{\hat{r}} = \sqrt{1 - b/r}\,\partial_r \,, \quad \xi_{\hat{\theta}} = r^{-1}\partial_\theta \,, \quad \xi_{\hat{\phi}} = (r \sin\theta)^{-1}\partial_\phi \,, \tag{3.7}$$

and it turns out that the energy-momentum tensor has the form

$$T_{\mu\nu} = \rho \xi_{\hat{t}\mu}\xi_{\hat{t}\nu} + p_r \xi_{\hat{r}\mu}\xi_{\hat{r}\nu} + p_l(\xi_{\hat{\theta}\mu}\xi_{\hat{\theta}\nu} + \xi_{\hat{\phi}\mu}\xi_{\hat{\phi}\nu}) \,, \tag{3.8}$$

where the rest energy density ρ, the radial pressure p_r and the lateral pressure p_l are given by

$$\rho = \frac{b'}{8\pi G r^2} \,, \tag{3.9a}$$

$$p_r = \frac{1}{8\pi G}\left[2\left(1 - \frac{b}{r}\right)\frac{\Phi'}{r} - \frac{b}{r^3}\right] , \tag{3.9b}$$

$$p_l = \frac{1}{8\pi G}\left[\left(1 - \frac{b}{r}\right)\left(\Phi'' + \Phi'^2 + \frac{\Phi'}{r}\right) - \frac{b'r - b}{2r^2}\left(\Phi' + \frac{1}{r}\right)\right] . \tag{3.9c}$$

We now have a look at the null energy condition in the form (2.13). In particular we consider the radial pressure. One can check that

$$\rho + p_r = -\frac{e^{2\Phi}}{r}\frac{d}{dr}\left[e^{-2\Phi}\left(1 - \frac{b}{r}\right)\right] = -\frac{e^{2\Phi}}{r}F'(r) \,. \tag{3.10}$$

At the wormhole throat $r = r_0$ we have $F(r_0) = 0$, since $b(r_0) = r_0$. Anywhere else we have $F(r) > 0$ for $r > r_0$. From this follows that $F'(r) > 0$ for some region near the throat. But this means that $\rho + p_r < 0$, and so the null energy condition is violated. From the relations between the different energy conditions it follows that also all other energy conditions are violated. Traversable wormholes therefore require exotic matter.

4 Warp drives

Another example for an exotic solution of the Einstein equations is the warp drive, which is given by the metric [6, 7]

$$ds^2 = -dt^2 + dx^2 + dy^2 + \left[dz - v(t) f\left(\sqrt{x^2 + y^2 + (z - z_0(t))^2}\right) dt\right]^2 , \tag{4.1}$$

where $v(t) = z_0'(t)$ and

$$f(r) = \frac{\tanh[\sigma(r + R)] - \tanh[\sigma(r - R)]}{2\tanh(\sigma R)} \tag{4.2}$$

with constants $R > 0$, $\sigma > 0$. This spacetime metric describes a "bubble" of radius R and inverse wall thickness σ which "slides" through spacetime with a velocity $v(t)$, where the center of the bubble is at $x(t) = 0, y(t) = 0, z(t) = z_0(t)$. An observer who stays in the center of the bubble moves on a timelike geodesic and his clock measures the same time as a clock outside the bubble, in the asymptotically flat surrounding spacetime. The velocity of the bubble can be arbitrarily high, allowing the observer to travel arbitrary distances in finite time.

The calculation of the energy-momentum tensor for the warp drive metric is quite lengthy. We thus look at only one component of the energy-momentum tensor, which describes the energy measured by a co-moving observer, i.e., an observer whose four-velocity ξ^μ is given by

$$\xi = \partial_t + vf\partial_z \,. \tag{4.3}$$

This observer measures the energy density

$$T_{\mu\nu}\xi^\mu\xi^\nu = -\frac{1}{32\pi G}\frac{v^2(x^2+y^2)}{r^2}f'^2 < 0\,. \tag{4.4}$$

This clearly violates the weak energy condition (1.1).

References

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