Selected Topics in the Theories of Gravity

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1 Definitions

1.1 Isometries

Let (M, g) be a Riemannian manifold and $f : M \to M, x \mapsto f(x) = x'$ a diffeomorphism, i.e., a bijective map such that both f and its inverse f^{-1} are differentiable. Under this diffeomorphism the metric changes to

$$g'_{\mu\nu}(x') = g_{\rho\sigma}(x)\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\,. \tag{1.1}$$

We now compare this transformed metric $g'_{\mu\nu}$ to the original metric $g_{\mu\nu}$ at the same spacetime point x'. This comparison makes sense, since both are tensors at the same point x'. A diffeomorphism f which leaves the metric invariant, $g_{\mu\nu}(x') = g'_{\mu\nu}(x')$, is called an *isometry*. If f is an isometry, then also its inverse f^{-1} is an isometry. If f_1, f_2 are isometries, then also their composition $f_1 \circ f_2$ is an isometry. All isometries of a given Riemannian manifold (M, g) therefore form a group, called the *isometry group* of (M, g).

1.2 Orbits

Let G be a group of diffeomorphisms of a manifold M. For every point $x \in M$ we define the orbit of G through M as

$$O_x = \{f(x) | f \in G\}\,. \tag{1.2}$$

Every point belongs to exactly one orbit: for two points x, x' the orbits are either disjoint, $O_x \cap O_{x'} = \emptyset$, or identical, $O_x = O_{x'}$.

1.3 Killing vector fields

Here we are in particular interested in isometry groups which are Lie groups. In this case we can pick a 1-parameter subgroup, which is a family $(f_u, u \in \mathbb{R})$ of diffeomorphisms such that $f_u \circ f_v = f_{u+v}$. For sufficiently small u we can perform a Taylor expansion and consider the infinitesimal diffeomorphism

$$x'^\mu = x^\mu + u\xi^\mu(x) + \mathcal{O}(u^2) \tag{1.3}$$

with a vector field ξ^μ given by

$$\xi(x) = \left.\frac{\partial f_u(x)}{\partial u}\right|_{u=0}\,. \tag{1.4}$$

The change of the metric at a point x' is then given by

$$\begin{aligned}
g_{\mu\nu}(x') - g'_{\mu\nu}(x') &= g_{\mu\nu}(x + u\xi) - g_{\rho\sigma}(x)\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} + \mathcal{O}(u^2) \\
&= g_{\mu\nu}(x) + u\xi^\rho\partial_\rho g_{\mu\nu}(x) - g_{\rho\sigma}(x)\left(\delta^\rho_\mu - u\partial_\mu\xi^\rho\right)\left(\delta^\sigma_\nu - u\partial_\nu\xi^\sigma\right) + \mathcal{O}(u^2) \\
&= u\xi^\rho\partial_\rho g_{\mu\nu} + u(\partial_\mu\xi^\rho)g_{\rho\nu} + u(\partial_\nu\xi^\sigma)g_{\mu\sigma} + \mathcal{O}(u^2) \\
&= u\mathcal{L}_\xi g_{\mu\nu} + \mathcal{O}(u^2)\,.
\end{aligned} \tag{1.5}$$

Since f_u is an isometry it follows that $\mathcal{L}_\xi g_{\mu\nu} = 0$. A vector field which satisfies this property is called a *Killing vector field.* Any linear combination of Killing vector fields (with constant coefficients) is again a Killing vector field. Also the Lie bracket $[\xi, \xi']$ of two Killing vector fields is again a Killing vector field. All Killing vector fields of a given Riemannian manifold together with the Lie bracket thus form a Lie algebra. This is the Lie algebra of the isometry group. All Killing vector fields are tangent to the orbits of the isometry group.

2 Example: planar symmetry

2.1 Isometries

Consider the metric given by

$$ds^2 = -A(t,z)dt^2 + B(t,z)dz^2 + C(t,z)(dx^2 + dy^2) = g_{\mu\nu}(t,x,y,z)dx^\mu dx^\nu \tag{2.1}$$

in Cartesian coordinates (t, x, y, z) and the diffeomorphism $f_{\phi,c_x,c_y} : x \mapsto x'$ given by

$$t' = t\,, \quad z' = z\,, \quad x' = x\cos\phi + y\sin\phi + c_x\,, \quad y' = -x\sin\phi + y\cos\phi + c_y \tag{2.2}$$

with parameters ϕ, c_x, c_y. This diffeomorphism describes a rotation in the (x, y)-plane by the angle ϕ, followed by a translation by the shift vector (c_x, c_y). The inverse of this diffeomorphism is a shift by $(-c_x, -c_y)$, followed by a rotation by $-\phi$, and thus given by

$$t = t'\,, \quad z = z'\,, \quad x = (x' - c_x)\cos\phi - (y' - c_y)\sin\phi\,, \quad y = (x' - c_x)\sin\phi + (y' - c_y)\cos\phi\,. \tag{2.3}$$

Its differential is then given by

$$dt = dt'\,, \quad dz = dz'\,, \quad dx = dx'\cos\phi - dy'\sin\phi\,, \quad dy = dx'\sin\phi + dy'\cos\phi\,. \tag{2.4}$$

With these expressions we can calculate the transformation of the line element, which takes the form

$$\begin{aligned}
ds^2 &= -A(t',z')dt'^2 + B(t',z')dz'^2 + C(t',z')\left[(dx'\cos\phi - dy'\sin\phi)^2 + (dx'\sin\phi + dy'\cos\phi)^2\right] \\
&= -A(t',z')dt'^2 + B(t',z')dz'^2 + C(t',z')(dx'^2 + dy'^2) \\
&= g'_{\mu\nu}(t',x',y',z')dx'^\mu dx'^\nu\,.
\end{aligned} \tag{2.5}$$

Comparing the metric coefficients we now see that $g'_{\mu\nu}(t', x', y', z') = g_{\mu\nu}(t', x', y', z')$, so that f is indeed an isometry. The isometry group is isomorphic to the Euclidean group $\mathrm{E}(2) = \mathrm{ISO}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$.

2.2 Orbits

If we fix a point (t, x, y, z) and apply an arbitrary group element f_{ϕ,c_x,c_y}, we see that the image has always $t' = t$ and $z' = z$, independent of the choice of f_{ϕ,c_x,c_y}. On the contrary, we can always choose some f_{ϕ,c_x,c_y} for any values of x' and y'. The orbit of the isometry group through (t, x, y, z) is thus the plane $\{(t', x', y', z')|t' = t, z' = z\}$.

2.3 Killing vector fields

We now consider the 1-parameter subgroup $f_\phi = f_{\phi,0,0}$ of pure rotations in the (x, y)-plane around $(0, 0)$. This is indeed a 1-parameter subgroup, since from

$$x' = x\cos\phi + y\sin\phi\,, \quad y' = -x\sin\phi + y\cos\phi \tag{2.6}$$

and

$$x'' = x'\cos\phi' + y'\sin\phi'\,, \quad y'' = -x'\sin\phi' + y'\cos\phi' \tag{2.7}$$

follows

$$x'' = x\cos(\phi+\phi') + y\sin(\phi+\phi')\,, \quad y'' = -x\sin(\phi+\phi') + y\cos(\phi+\phi')\,, \tag{2.8}$$

so that $f_{\phi'} \circ f_\phi = f_{\phi'+\phi}$. It thus follows that

$$\xi_r(x) = \left.\frac{\partial f_\phi(t,x,y,z)}{\partial\phi}\right|_{\phi=0} = y\partial_x - x\partial_y \tag{2.9}$$

is a Killing vector field associated to rotations around $(0, 0)$. Similarly, the 1-parameter subgroups $f_{0,c_x,0}$ and $f_{0,0,c_y}$ of translation correspond to the Killing vector fields $\xi_x = \partial_x$ and $\xi_y = \partial_y$.
Because of the translation invariance there is nothing special about the point $(0, 0)$. The metric is invariant under rotation around any point (x_0, y_0). The corresponding 1-parameter subgroup is then simply given by

$$x' = (x-x_0)\cos\phi + (y-y_0)\sin\phi + x_0\,, \quad y' = -(x-x_0)\sin\phi + (y-y_0)\cos\phi + y_0\,. \tag{2.10}$$

The Killing vector field is then

$$\xi = (y-y_0)\partial_x - (x-x_0)\partial_y = \xi_r - y_0\xi_x + x_0\xi_y\,, \tag{2.11}$$

which is simply a linear combination of the Killing vector fields we derived.

3 Catalog of symmetries

3.1 Stationary spacetimes

A stationary spacetime is a spacetime which has a timelike Killing vector field ξ. One can choose coordinates (t, x^i) so that $\xi = \partial_t$ and the metric has the form

$$ds^2 = -\lambda(dt - \omega_i dx^i)^2 + h_{ij}dx^i dx^j\,, \tag{3.1}$$

where $\lambda, \omega_i, h_{ij}$ are free functions of the spatial coordinates x^i.

3.2 Static spacetimes

A static spacetime is a stationary spacetime whose timelike Killing vector field is hypersurface-orthogonal. This restricts the metric further to

$$ds^2 = -\lambda dt^2 + h_{ij} dx^i dx^j \,, \tag{3.2}$$

where λ, h_{ij} are free functions of the spatial coordinates x^i.

3.3 Axisymmetric spacetimes

An axisymmetric spacetime is a spacetime whose isometry group contains a subgroup isomorphic to SO(2), whose orbits are circles around a set of fixed points (the axis of rotation). One can choose coordinates such that one of them, which we denote ϕ, is 2π-periodic and parametrizes the orbits. The metric has the general form

$$ds^2 = \lambda(d\phi - \omega_i dx^i)^2 + h_{ij} dx^i dx^j \,, \tag{3.3}$$

where $\lambda, \omega_i, h_{ij}$ are free functions of the remaining three coordinates x^i.

3.4 Spherically symmetric spacetimes

An spherically symmetric spacetime is a spacetime whose isometry group contains a subgroup isomorphic to SO(3), whose orbits are spheres around a set of fixed points (the center of rotation). The metric in spherical coordinates takes the general form

$$ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + C(t,r)(d\theta^2 + \sin^2\theta d\phi^2) \,. \tag{3.4}$$

3.5 Homogeneous, isotropic spacetimes

A homogeneous, isotropic spacetime has six linearly independent Killing vector fields, three of them describing rotations and three describing translations. The metric has the general form

$$ds^2 = -n^2(t)dt^2 + a^2(t)d\Sigma^2 \,, \tag{3.5}$$

where the lapse function $n(t)$ can always be set to $n(t) \equiv 1$ by a rescaling of the time coordinate, and $d\Sigma$ denotes the line element of a maximally symmetric 3-dimensional space. There are only three different of these spaces:

- The 3-sphere with isometry group SO(4):

$$d\Sigma^2 = \frac{dr^2}{1 - r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \,. \tag{3.6a}$$

- Euclidean 3-space with isometry group $\mathrm{E}(3) = \mathrm{ISO}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$:

$$d\Sigma^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \,. \tag{3.6b}$$

- Hyperbolic 3-space with isometry group $\mathrm{SO}(3,1)$:

$$d\Sigma^2 = \frac{dr^2}{1 + r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \,. \tag{3.6c}$$

3.6 Maximally symmetric spacetimes

A maximally symmetric spacetime has the maximal number of ten linearly independent Killing vector fields, which fixes the metric completely (up to coordinate transformations). There are only three different maximally symmetric spacetimes:

- De Sitter space with isometry group SO$(4,1)$:

$$ds^2 = -(1-r^2)dt^2 + \frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{3.7a}$$

- Minkowski space with isometry group ISO$(3,1) = $ SO$(3,1) \ltimes \mathbb{R}^{3,1}$:

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{3.7b}$$

- Anti-de Sitter space with isometry group SO$(3,2)$:

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{3.7c}$$