Selected Topics in the Theories of Gravity

Manuel Hohmann

10. March 2014

1 Cosmological symmetry

In cosmology we start from the assumption of a homogeneous, isotropic spacetime. It is characterized by six linearly independent Killing vector fields, three of them describing rotations and three describing translations. The metric has the general form

$$ds^2 = -n^2(t)dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j\,, \quad (1.1)$$

where we set the lapse function $n(t)$ to $n(t) \equiv 1$ by a choice of the time coordinate and the dynamics is entirely contained in the time dependent scale factor $a(t)$. The spatial geometry is determined by the constant curvature parameter k in the spatial metric γ_{ij}, which corresponds to one of the three maximally symmetric 3-spaces:

- $k = 1$: The 3-sphere with isometry group SO(4).
- $k = 0$: Euclidean 3-space with isometry group $\mathrm{E}(3) = \mathrm{ISO}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$.
- $k = -1$: Hyperbolic 3-space with isometry group $\mathrm{SO}(3,1)$.

The determinant of the metric is given by

$$g = -a^6\gamma\,, \quad (1.2)$$

where γ is the determinant of the spatial metric γ_{ij}. The non-vanishing Christoffel symbols are given by

$$\Gamma^0{}_{ij} = a\dot{a}\gamma_{ij}\,, \quad \Gamma^i{}_{0i} = \frac{\dot{a}}{a}\delta^i_j, \quad \Gamma^k{}_{ij} = \Gamma^k{}_{ij}(\gamma)\,, \quad (1.3)$$

where $\Gamma^k{}_{ij}(\gamma)$ are the Christoffel symbols of the spatial metric γ_{ij} and dots denote derivatives with respect to t. The Riemann tensor is given by

$$R^0{}_{i0j} = a\ddot{a}\gamma_{ij}\,, \quad R^i{}_{0j0} = -\frac{\ddot{a}}{a}\delta^i_j\,, \quad R^i{}_{jkl} = \left(k + \dot{a}^2\right)\gamma^{im}(\gamma_{mk}\gamma_{jl} - \gamma_{ml}\gamma_{jk})\,. \quad (1.4)$$

The Ricci tensor takes the form

$$R_{00} = -3\frac{\ddot{a}}{a}\,, \quad R_{0i} = 0\,, \quad R_{ij} = (2k + 2\dot{a}^2 + a\ddot{a})\gamma_{ij} \quad (1.5)$$

and the Ricci scalar

$$R = 6\frac{k + \dot{a}^2 + a\ddot{a}}{a^2}\,. \quad (1.6)$$

2 Cosmological dynamics

The dynamics are given by the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \,. \tag{2.1}$$

It follows from the results of the previous section that the energy-momentum tensor must be of the perfect fluid form

$$T_{00} = \rho \,, \quad T_{0i} = 0 \,, \quad T_{ij} = p a^2 \gamma_{ij} \tag{2.2}$$

with a homogeneous and isotropic energy density $\rho = \rho(t)$ and pressure p. Note that this perfect fluid form is not an assumption, but a consequence of the assumed cosmological symmetry. The field equations then reduce to

$$3 \frac{k + \dot{a}^2}{a^2} - \Lambda = 8\pi G \rho \,, \tag{2.3a}$$

$$-\frac{k + \dot{a}^2 + 2a\ddot{a}}{a^2} + \Lambda = 8\pi G p \,. \tag{2.3b}$$

The second equation can be solved for $\ddot{a}$, which yields

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \,. \tag{2.4}$$

Equations (2.3a) and (2.4) are the Friedmann equations. Equation (2.4) can also be replaced by

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) \,, \tag{2.5}$$

which corresponds to the continuity equation $\nabla_\mu T^{\mu 0} = 0$ and can be obtained from taking the time derivative of equation (2.3a). In order to solve these differential equations it is often useful to use the cosmological time parameter η defined by $dt = a\, d\eta$ instead of t, from which one obtains

$$3\left(\frac{a'^2}{a^4} + \frac{k}{a^2}\right) - \Lambda = 8\pi G \rho \,, \tag{2.6a}$$

$$-2\frac{a''}{a^3} + \frac{a'^2}{a^4} - \frac{k}{a^2} + \Lambda = 8\pi G p \,. \tag{2.6b}$$

Here primes denote derivatives with respect to η. In order to solve these equations one further needs to specify a relation between density ρ and pressure p. This relation depends on the matter content. Note that the cosmological constant has the same effect as a perfect fluid with constant density and pressure

$$\rho = \frac{\Lambda}{8\pi G} \,, \quad p = -\frac{\Lambda}{8\pi G} \,. \tag{2.7}$$

In the following we will therefore neglect the cosmological constant and instead model it by an appropriate contribution to the matter content of the universe.

3 Classical solutions

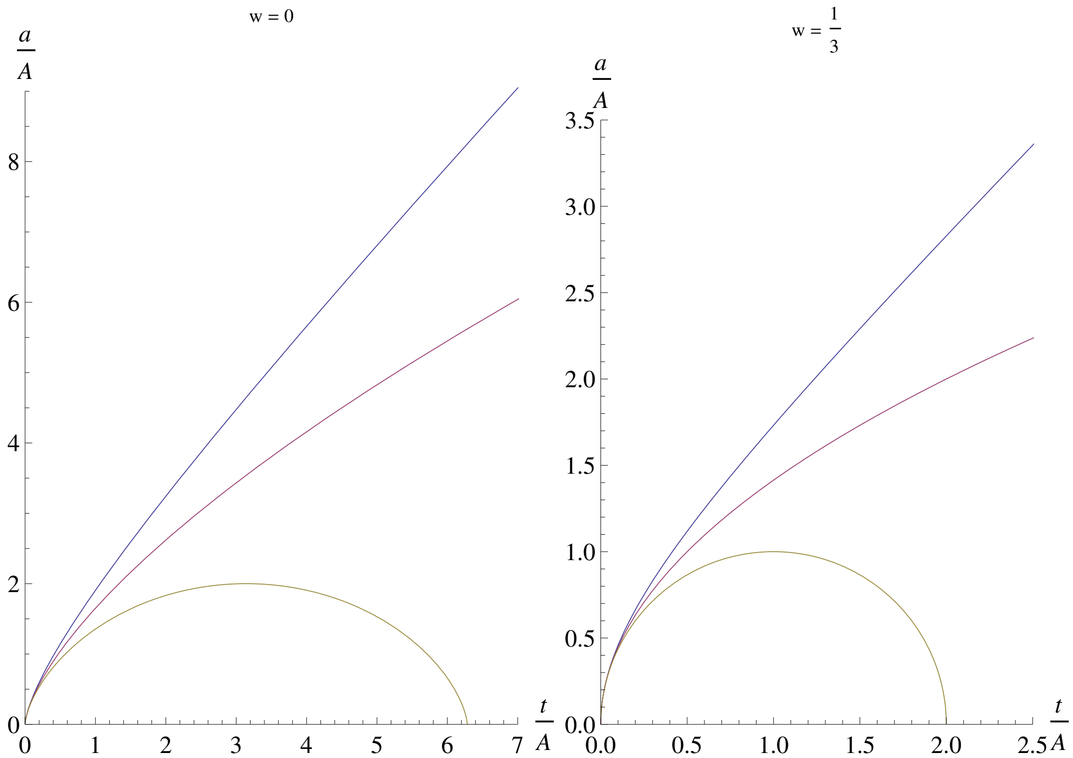
For the classical solutions one assumes an equation of state of the form

$$p = w\rho\,, \tag{3.1}$$

where the constant barotropic index w is $w = 0$ for dust or $w = 1/3$ for radiation. For $\Lambda = 0$ this leads to the following solutions with constant scale parameter A:

	$w = 0$		$w = \frac{1}{3}$	
$k = -1$	$a = A(\cosh\eta - 1)$,	$t = A(\sinh\eta - \eta)$	$a = A\sinh\eta$,	$t = A(\cosh\eta - 1)$
$k = 0$	$a = \frac{A}{2}\eta^2$,	$t = \frac{A}{6}\eta^3$	$a = A\eta$,	$t = \frac{A}{2}\eta^2$
$k = 1$	$a = A(1 - \cos\eta)$,	$t = A(\eta - \sin\eta)$	$a = A\sin\eta$,	$t = A(1 - \cos\eta)$

All solutions have a Big Bang singularity with $a = 0$ at the beginning $t = 0$ of time, where the expansion of the universe begins. This expansion slows down as the universe evolves, and for $k = 1$ even stops and leads to a collapse of the universe. For $k = -1$ the universe continues to expand forever. The critical solution between these two is the $k = 0$ solution, where the universe also keeps expanding, but the expansion rate tends to 0. The model for dust matter with $k = 0$ is also known as the Einstein-de Sitter model.

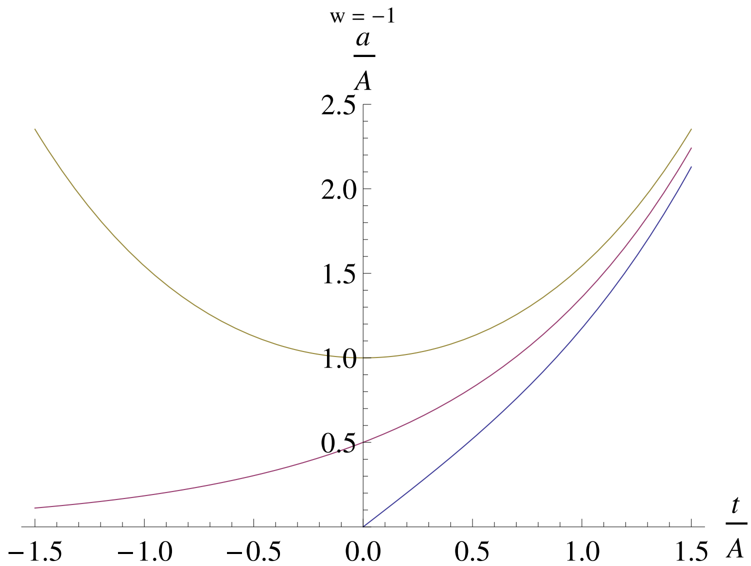


4 Dark energy and the cosmological constant

Note that in the models with $w = 0$ and $w = 1/3$ the acceleration (2.4) is always negative, since $\rho + 3p$ is positive. In order to obtain an accelerating expansion one need matter with a barotropic index $w < -1/3$, which is known as dark energy. A positive cosmological constant, which corresponds to dark energy with $w = -1$, satisfies this condition. If we assume no other matter to be present, we obtain the following solutions with $\eta < 0$:

$$
\begin{array}{l|ll}
k=-1 & a=-A/\sinh\eta, & t=-A\ln(-\tanh\frac{\eta}{2}) \\
k=0 & a=-A/\eta, & t=-A\ln(-\frac{\eta}{2}) \\
k=1 & a=-A/\sin\eta, & t=-A\ln(-\tan\frac{\eta}{2})
\end{array}
$$

For $k=-1$ the universe starts from a Big Bang singularity, while for $k=0$ and $k=-1$ it extends into the infinite past. For $k=0$ the expansion is exponential, $a\sim e^{t/A}$, so that in the infinite past the universe becomes infinitely small. This model is also known as the de Sitter model. For $k=1$ there is a positive minimal radius at $t=0$. In all solutions the universe expands forever.



5 Practical cosmology

While the variables we used in the previous sections are helpful for understanding the geometry of the universe and its evolution, they are less helpful for experimental purposes. In cosmological experiments one typically observes physical quantities at a fixed time, such as the present time or the time of the generation of the cosmic microwave background. An important observable is the Hubble parameter

$$H=\frac{\dot{a}}{a}\,, \tag{5.1}$$

which can be measured from the redshift and distance of different galaxies. Its present value is denoted by H_0. In contrast, the present value of the scale factor a_0 cannot be measured directly.

We have seen that the evolution of the universe crucially depends on the choice of the curvature parameter k. In order to determine this parameter one make use of the first Friedmann equation (2.3a), where we again set $\Lambda=0$ and include its effect into an appropriate contribution to the matter content. If the density ρ is smaller than the critical density

$$\rho_{\text{crit}}=\frac{3H^2}{8\pi G}\,, \tag{5.2}$$

the term k/a^2 must be negative, so that $k=-1$. Correspondingly, if the density is larger that the critical density, it follows that $k=1$. Finally, from $\rho=\rho_{\text{crit}}$ follows $k=0$.

From the present time values ρ_0 and H_0 of the density and the Hubble parameter one defines the ratio

$$\Omega = \frac{8\pi G\rho_0}{3H_0^2}, \tag{5.3}$$

which is a constant since it depends only on quantities measured at the fixed present time. Using the continuity equation (2.5) one finds that the time dependent density is given by

$$\rho = \rho_0\left(\frac{a_0}{a}\right)^{3w+3} = \Omega\frac{3H_0^2}{8\pi G}\left(\frac{a_0}{a}\right)^{3w+3} \tag{5.4}$$

for a perfect fluid with barotropic index w. However, in practice one usually assumes a mixture of different matter components. In the ΛCDM model, where one has a cosmological constant and dark matter in addition to the ordinary baryonic matter and radiation, these components have densities ρ_Λ with $w = -1$, ρ_M with $w = 0$ and ρ_R with $w = 1/3$. For the present time one thus defines

$$\Omega_\Lambda = \frac{8\pi G\rho_{\Lambda 0}}{3H_0^2}, \quad \Omega_M = \frac{8\pi G\rho_{M0}}{3H_0^2}, \quad \Omega_R = \frac{8\pi G\rho_{R0}}{3H_0^2}, \quad \Omega = \Omega_\Lambda + \Omega_M + \Omega_R. \tag{5.5}$$

The first Friedmann equation (2.3a) at present time then yields the relation

$$\Omega_\Lambda + \Omega_M + \Omega_R = 1 + \frac{k}{a_0^2H_0^2} = 1 - \Omega_K, \tag{5.6}$$

where one defines

$$\Omega_K = -\frac{k}{a_0^2H_0^2}. \tag{5.7}$$

The sign of Ω_K then determines the value of k. Current observations that Ω_K is small, so that one often encounters the assumption $k = 0$ of a flat universe.