Gauge-invariant higher order perturbation theory

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1 Definition of gauge

We start our discussion of gauge invariance with a general remark on the use of gauging in the literature. In the context of diffeomorphism invariance, the term *gauge* is often used synonymously to denote a choice of coordinates, hence a chart of the spacetime manifold, which is then used to express the components of tensor fields. Following this interpretation, gauge transformations are represented by changes of coordinates. This corresponds to what is known as passive interpretation of a diffeomorphism: points on the manifold and tensor fields at these points are the same, but the labels given to these points and the tensor components with respect to this labeling change. For our purposes, however, it will turn out to be more convenient to resort to the active interpretation of diffeomorphisms: a fixed coordinate system is chosen, points are mapped to a different position and tensor fields are moved and changed along with them. We will make this notion mathematically precise below, following the definitions given in [BS99, Nak05].
Let $\bar{M}$ be a manifold equipped with some tensor field $\bar{A}$, which we will regard as the background around which we will consider perturbations. In the context of gravity theories, this tensor field will typically be a metric $\bar{g}$, but one may consider also other / additional tensor fields; in this case we call $(\bar{M}, \bar{g})$ the background spacetime, and $\bar{g}$ the background metric. We usually choose this background spacetime to be some "standard" spacetime equipped with a fixed choice of coordinates $(x^\mu : \bar{M} \to \mathbb{R})$. Typically this metric will be a highly symmetric, exact solution to the gravitational field equations of the gravity theory we study. Common examples include:

1. maximally symmetric spacetimes, such as Minkowski, de Sitter or anti-de Sitter spacetimes;
2. the cosmologically symmetric Friedmann-Lemaître-Robertson-Walker spacetime;
3. spherically symmetric spacetimes, corresponding to non-rotating black holes or other compact objects;
4. axially symmetric spacetimes, describing rotating black holes or other compact objects.

The background spacetime will serve as a reference, to which we can compare a second, different, but diffeomorphic manifold M equipped with a tensor field A of the same signature as $\bar{A}$. In the case of a background metric $\bar{g}$, we thus have a metric g, which we call the physical spacetime and physical metric, respectively. However, despite being of the same signature, we cannot immediately compare the two tensor fields A and $\bar{A}$ for two reasons:

1. The tensor fields A and $\bar{A}$ are defined on two different manifolds. There is no notion of a "background contribution" of A on the physical spacetime M, since the background $\bar{A}$ is defined only on the reference spacetime $\bar{M}$.
2. There is no canonical identification between points of the physical and reference spacetimes. In other words, there is no canonical choice of coordinates on the physical spacetime M.

In order to compare the two tensor fields, we must therefore choose a diffeomorphism $\mathcal{X} : \bar{M} \to M$. This diffeomorphism will do two things:

1. It allows us to identify points on the manifolds $\bar{M}$ and M. Hence, it will equip M with a distinguished choice of coordinates $(\underset{\mathcal{X}}{x}^\mu : M \to \mathbb{R})$, obtained from the coordinates on $\bar{M}$ as $\underset{\mathcal{X}}{x}^\mu = x^\mu \circ \mathcal{X}^{-1}$.

2. It defines the pullback $\mathcal{X}^*A$ of the tensor field A to $\bar{M}$, which we will also write as $\underset{\mathcal{X}}{A}$, and which we can compare to the background field $\bar{A}$.

The diffeomorphism $\mathcal{X}$ is what we will call a *gauge* [BS99, Nak05]. Note that there is no canonical choice for such a diffeomorphism.

2 Perturbation theory

In perturbation theory we assume that the physical field depends on a parameter ϵ, commonly called the perturbation parameter, and so we will denote it by A_ϵ. We also assume that for each value of ϵ, the tensor field A_ϵ is defined on a different physical spacetime M_ϵ. In order to compare the physical and background fields, we therefore need a family of diffeomorphisms $\mathcal{X}_\epsilon : \bar{M} \to M_\epsilon$, which relate the physical and background spacetimes. This family is what we will call a gauge in the context of perturbation theory. Further, we assume that the "unperturbed" field A_0 on A_0 agrees with the background field $\bar{A}$ on $\bar{M}$. Hence, for consistency we assume that $M_0 = \bar{M}$ is the background spacetime itself, and that $\mathcal{X}_0 = \mathrm{id}_{\bar{M}} : \bar{M} \to \bar{M}$ is the identity map.[1]
A key idea of perturbation theory is the assumption that the physical field A_ϵ can be approximated by a series expansion in the perturbation parameter ϵ, whose zeroth order is the background field $\bar{A}$. Following our discussion above, we see that we cannot perform such a series expansion directly, since A_ϵ and $\bar{A}$ are defined on different manifolds. We can, however, express the pullback $\underset{\mathcal{X}}{A}_\epsilon = \mathcal{X}_\epsilon^* A_\epsilon$, which is defined on $\bar{M}$, as a series expansion of the form

$$\underset{\mathcal{X}}{A}_\epsilon = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \frac{\mathrm{d}^k}{\mathrm{d}\epsilon^k} \underset{\mathcal{X}}{A}_\epsilon \bigg|_{\epsilon=0} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \underset{\mathcal{X}}{\overset{k}{A}} . \tag{2.1}$$

Clearly, $\underset{\mathcal{X}}{\overset{0}{A}} = \bar{A}$ is the background field, and hence independent of the choice of the gauge $\mathcal{X}_\epsilon$. The other series coefficients $\underset{\mathcal{X}}{\overset{k}{A}}$ for $k \geq 1$, however, will depend on the choice of the gauge. It is therefore important to understand the nature of this dependence.

3 Change of gauge

We now consider two different gauges $\mathcal{X}_\epsilon, \mathcal{Y}_\epsilon : \bar{M} \to M_\epsilon$. This allows us to construct a family $\Phi_\epsilon : \bar{M} \to \bar{M}$ of diffeomorphisms given by $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$. Note that Φ is only a one-parameter *family* of diffeomorphisms, but in general not a one-parameter *group*; one has $\Phi_{\epsilon+\epsilon'} \neq \Phi_\epsilon \circ \Phi_{\epsilon'}$ and $\Phi_{-\epsilon} \neq \Phi_\epsilon^{-1}$ in general. Hence, it is not infinitesimally generated by a vector field. Nevertheless, one can obtain an infinitesimal, perturbative description of Φ as follows [SB98]. Consider a smooth function $f : \bar{M} \to \mathbb{R}$, and its pullback $\Phi_\epsilon^* f = f \circ \Phi_\epsilon$. For every $x \in \bar{M}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} (\Phi_\epsilon^* f)(x) \bigg|_{\epsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\epsilon} f(\Phi_\epsilon(x)) \bigg|_{\epsilon=0} = \overset{1}{\xi}(x) f , \tag{3.1}$$

where $\overset{1}{\xi}(x)$ is a derivation at x, since it is obviously linear and satisfies the Leibniz rule

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\epsilon} (\Phi_\epsilon^*(fg))(x) \bigg|_{\epsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} [f(\Phi_\epsilon(x)) g(\Phi_\epsilon(x))] \bigg|_{\epsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} f(\Phi_\epsilon(x)) \bigg|_{\epsilon=0} g(\Phi_0(x)) + f(\Phi_0(x)) \frac{\mathrm{d}}{\mathrm{d}\epsilon} g(\Phi_\epsilon(x)) \bigg|_{\epsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} (\Phi_\epsilon^* f)(x) \bigg|_{\epsilon=0} g(x) + f(x) \frac{\mathrm{d}}{\mathrm{d}\epsilon} (\Phi_\epsilon^* g)(x) \bigg|_{\epsilon=0} . \end{aligned} \tag{3.2}$$

Hence, it defines a tangent vector, which can explicitly be written as the tangent vector

$$\overset{1}{\xi}{}^\mu(x) = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \gamma_x^\mu(\epsilon) \bigg|_{\epsilon=0} \tag{3.3}$$

[1] Note that in contrast to the treatment in [Nak05, Nak07, Nak06] we will not regard the manifolds M_ϵ as leaves of a foliation of a manifold $N \cong M \times \mathbb{R}$.

to the curve

$$\epsilon \mapsto \gamma_x(\epsilon) = \Phi_\epsilon(x)\,. \tag{3.4}$$

The assignment $x \mapsto \overset{1}{\xi}(x)$ defines a vector field $\overset{1}{\xi}$ on $\bar{M}$, which defines a one-parameter group $\overset{1}{\phi}$, such that the curve

$$\epsilon \mapsto \overset{1}{\gamma}_x(\epsilon) = \overset{1}{\phi}_\epsilon(x) \tag{3.5}$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\overset{1}{\gamma}_x(\epsilon) = \overset{1}{\xi}(\overset{1}{\gamma}_x(\epsilon))\,. \tag{3.6}$$

Clearly, at $\epsilon = 0$, and in general only there, the curves γ_x and $\overset{1}{\gamma}_x$ have the same tangent vector. If Φ would constitute a one-parameter group, we would now simply have $\Phi = \overset{1}{\phi}$. However, in general this is not the case, and we can consider $\overset{1}{\phi}$ as an approximation to Φ at linear order in ϵ, since both generate the same linear order Taylor coefficient

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}(\Phi^*_\epsilon f)(x)\bigg|_{\epsilon=0} = \overset{1}{\xi}(x) f = \frac{\mathrm{d}}{\mathrm{d}\epsilon}(\overset{1}{\phi}{}^*_\epsilon f)(x)\bigg|_{\epsilon=0} \tag{3.7}$$

in a Taylor series for any function f. Omitting the argument x, we can write

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}(\Phi^*_\epsilon f)\bigg|_{\epsilon=0} = \mathcal{L}_{\overset{1}{\xi}} f = \frac{\mathrm{d}}{\mathrm{d}\epsilon}(\overset{1}{\phi}{}^*_\epsilon f)\bigg|_{\epsilon=0} \tag{3.8}$$

We can describe the deviation of Φ from a one-parameter group by defining a new one-parameter family of diffeomorphisms by

$$\Phi_\epsilon \circ \overset{1}{\phi}_{-\epsilon}\,. \tag{3.9}$$

For any smooth function $f : \bar{M} \to \mathbb{R}$ and $x \in \bar{M}$ one now easily checks that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}[(\Phi_\epsilon \circ \overset{1}{\phi}_{-\epsilon})^* f](x)\bigg|_{\epsilon=0} = 0\,. \tag{3.10}$$

Further, for the second derivative one finds that

$$\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}[(\Phi_\epsilon \circ \overset{1}{\phi}_{-\epsilon})^* f](x)\bigg|_{\epsilon=0} = \frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2} f(\Phi_\epsilon(\overset{1}{\phi}_{-\epsilon}(x)))\bigg|_{\epsilon=0} = \overset{2}{\xi}(x) f \tag{3.11}$$

defines another derivation at x. Linearity is again trivial to show, while the Leibniz rule follows from

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}[(\Phi_\epsilon \circ \overset{1}{\phi}_{-\epsilon})^*(fg)](x)\bigg|_{\epsilon=0} &= \frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}[f(\Phi_\epsilon(\overset{1}{\phi}_{-\epsilon}(x)))g(\Phi_\epsilon(\overset{1}{\phi}_{-\epsilon}(x)))]\bigg|_{\epsilon=0} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2} f(\Phi_\epsilon(\overset{1}{\phi}_{-\epsilon}(x)))\bigg|_{\epsilon=0} g(\Phi_0(\overset{1}{\phi}_0(x))) \\
&\quad + 2\,\underbrace{\frac{\mathrm{d}}{\mathrm{d}\epsilon} f(\Phi_\epsilon(\overset{1}{\phi}_{-\epsilon}(x)))\bigg|_{\epsilon=0}}_{=0}\underbrace{\frac{\mathrm{d}}{\mathrm{d}\epsilon} g(\Phi_\epsilon(\overset{1}{\phi}_{-\epsilon}(x)))\bigg|_{\epsilon=0}}_{=0} \\
&\quad + f(\Phi_0(\overset{1}{\phi}_0(x)))\,\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2} g(\Phi_\epsilon(\overset{1}{\phi}_{-\epsilon}(x)))\bigg|_{\epsilon=0} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}[(\Phi_\epsilon \circ \overset{1}{\phi}_{-\epsilon})^* f](x)\bigg|_{\epsilon=0} g(x) + f(x)\,\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}[(\Phi_\epsilon \circ \overset{1}{\phi}_{-\epsilon})^* g](x)\bigg|_{\epsilon=0}\,.
\end{aligned} \tag{3.12}$$

This means that $\overset{2}{\xi}(x)$ is a tangent vector at x. Hence, via the assignment $x \mapsto \overset{2}{\xi}(x)$ this defines another vector field $\overset{2}{\xi}$, which generates a one-parameter group $\overset{2}{\phi}$ of diffeomorphisms on $\bar{M}$. With the help of this one-parameter group, one can obtain a better approximation to Φ, given by

$$\overset{2}{\phi}_{\epsilon^2/2} \circ \overset{1}{\phi}_\epsilon\,. \tag{3.13}$$

This now constitutes an approximation to second order in ϵ, in the sense

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}(\Phi_\epsilon^* f)\bigg|_{\epsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}[(\overset{2}{\phi}_{\epsilon^2/2} \circ \overset{1}{\phi}_\epsilon)^* f]\bigg|_{\epsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}(\overset{1}{\phi}{}_\epsilon^* f)\bigg|_{\epsilon=0} = \mathcal{L}_{\overset{1}{\xi}} f\,, \tag{3.14a}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}(\Phi_\epsilon^* f)\bigg|_{\epsilon=0} = \frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}[(\overset{2}{\phi}_{\epsilon^2/2} \circ \overset{1}{\phi}_\epsilon)^* f]\bigg|_{\epsilon=0} = \mathcal{L}_{\overset{2}{\xi}} f + \mathcal{L}_{\overset{1}{\xi}}^2 f\,. \tag{3.14b}$$

Continuing this scheme further, one can approximate Φ by an (in general infinite) series of one-parameter groups $\overset{k}{\phi}$ of diffeomorphisms, called a *knight diffeomorphism*, such that [BMMS97]

$$\Phi_\epsilon = \cdots \overset{k}{\phi}_{\epsilon^k/k!} \circ \cdots \circ \overset{2}{\phi}_{\epsilon^2/2} \circ \overset{1}{\phi}_\epsilon\,. \tag{3.15}$$

Since each $\overset{k}{\phi}$ is a one-parameter group of diffeomorphisms, it is generated by a vector field $\overset{k}{\xi}$. With the help of these vector fields, one can continue the Taylor expansion of $\Phi_\epsilon^* f$, and finds the formula

$$\frac{\mathrm{d}^k}{\mathrm{d}\epsilon^k}(\Phi_\epsilon^* f)\bigg|_{\epsilon=0} = \sum_{l_1+2l_2+\ldots=k} \frac{k!}{(1!)^{l_1}(2!)^{l_2}\cdots l_1! l_2! \cdots} \mathcal{L}_{\overset{1}{\xi}}^{l_1} \mathcal{L}_{\overset{2}{\xi}}^{l_2} \cdots f\,, \tag{3.16}$$

which is essentially a generalization of Faà di Bruno's formula for the chain rule applied to the k'th derivative. We have assumed a scalar function f for the construction above, but it is easy to see that the same formula can also be derived for an arbitrary tensor field instead.

4 Gauge transformation of tensor fields

Let $\mathcal{X}_\epsilon, \mathcal{Y}_\epsilon : \bar{M} \to M_\epsilon$ be two gauges as before. The tensor fields $\underset{\mathcal{X}}{A}_\epsilon = \mathcal{X}_\epsilon^* g, \underset{\mathcal{Y}}{A}_\epsilon = \mathcal{Y}_\epsilon^* g$ in the different gauges, which are now both defined on the background spacetime $\bar{M}$, are related by

$$\underset{\mathcal{Y}}{A}_\epsilon = \Phi_\epsilon^* \underset{\mathcal{X}}{A}_\epsilon\,. \tag{4.1}$$

Using the construction (3.15), one can now relate the perturbative expansions of A in the two gauges. Note that the relation (4.1) resembles the construction above, since we consider the pullback along Φ_ϵ. However, there is one difference arising from the fact that now also the tensor field $\underset{\mathcal{X}}{A}_\epsilon$ depends on ϵ. Hence, taking the Taylor series coefficients in ϵ, it must be expanded as well, using the formula (2.1). The derivatives with respect to ϵ are then distributed, and one finds the formula

$$\begin{aligned}
\underset{\mathcal{Y}}{\overset{k}{A}} &= \frac{\mathrm{d}^k}{\mathrm{d}\epsilon^k} \underset{\mathcal{Y}}{A}_\epsilon\bigg|_{\epsilon=0} \\
&= \frac{\mathrm{d}^k}{\mathrm{d}\epsilon^k} \Phi_\epsilon^* \underset{\mathcal{X}}{A}_\epsilon\bigg|_{\epsilon=0} \\
&= \sum_{0\le l_1+2l_2+\ldots\le k} \binom{k}{l_1+2l_2+\ldots} \frac{(l_1+2l_2+\ldots)!}{(1!)^{l_1}(2!)^{l_2}\cdots l_1! l_2!\cdots} \mathcal{L}_{\overset{1}{\xi}}^{l_1} \cdots \mathcal{L}_{\overset{j}{\xi}}^{l_j} \cdots \frac{\mathrm{d}^{k-l_1-2l_2-\ldots}}{\mathrm{d}\epsilon^{k-l_1-2l_2-\ldots}} \underset{\mathcal{Y}}{A}_\epsilon\bigg|_{\epsilon=0} \\
&= \sum_{0\le l_1+2l_2+\ldots\le k} \frac{k!}{(k-l_1-2l_2-\ldots)!(1!)^{l_1}(2!)^{l_2}\cdots l_1! l_2!\cdots} \mathcal{L}_{\overset{1}{\xi}}^{l_1} \cdots \mathcal{L}_{\overset{j}{\xi}}^{l_j} \cdots \underset{\mathcal{X}}{\overset{k-l_1-2l_2-\ldots}{A}}.
\end{aligned} \tag{4.2}$$

Writing out the lowest four orders of this formula we find

$$\underset{\mathcal{Y}}{\overset{0}{A}} = \underset{\mathcal{X}}{\overset{0}{A}} = \bar{A}\,, \tag{4.3a}$$

$$\underset{\mathcal{Y}}{\overset{1}{A}} = \underset{\mathcal{X}}{\overset{1}{A}} + \mathcal{L}_{\overset{1}{\xi}} \underset{\mathcal{X}}{\overset{0}{A}}\,, \tag{4.3b}$$

$$\underset{\mathcal{Y}}{\overset{2}{A}} = \underset{\mathcal{X}}{\overset{2}{A}} + 2\mathcal{L}_{\overset{1}{\xi}} \underset{\mathcal{X}}{\overset{1}{A}} + \mathcal{L}_{\overset{2}{\xi}} \underset{\mathcal{X}}{\overset{0}{A}} + \mathcal{L}_{\overset{1}{\xi}}^2 \underset{\mathcal{X}}{\overset{0}{A}}\,, \tag{4.3c}$$

$$\underset{\mathcal{Y}}{\overset{3}{A}} = \underset{\mathcal{X}}{\overset{3}{A}} + 3\mathcal{L}_{\overset{1}{\xi}} \underset{\mathcal{X}}{\overset{2}{A}} + 3\mathcal{L}_{\overset{2}{\xi}} \underset{\mathcal{X}}{\overset{1}{A}} + 3\mathcal{L}_{\overset{1}{\xi}}^2 \underset{\mathcal{X}}{\overset{1}{A}} + \mathcal{L}_{\overset{3}{\xi}} \underset{\mathcal{X}}{\overset{0}{A}} + 3\mathcal{L}_{\overset{1}{\xi}} \mathcal{L}_{\overset{2}{\xi}} \underset{\mathcal{X}}{\overset{0}{A}} + \mathcal{L}_{\overset{1}{\xi}}^3 \underset{\mathcal{X}}{\overset{0}{A}}\,, \tag{4.3d}$$

$$\underset{\mathcal{Y}}{\overset{4}{A}} = \underset{\mathcal{X}}{\overset{4}{A}} + 4\mathcal{L}_{\overset{1}{\xi}} \underset{\mathcal{X}}{\overset{3}{A}} + 6\mathcal{L}_{\overset{2}{\xi}} \underset{\mathcal{X}}{\overset{2}{A}} + 6\mathcal{L}_{\overset{1}{\xi}}^2 \underset{\mathcal{X}}{\overset{2}{A}} + 4\mathcal{L}_{\overset{3}{\xi}} \underset{\mathcal{X}}{\overset{1}{A}} + 12\mathcal{L}_{\overset{1}{\xi}} \mathcal{L}_{\overset{2}{\xi}} \underset{\mathcal{X}}{\overset{1}{A}} + 4\mathcal{L}_{\overset{1}{\xi}}^3 \underset{\mathcal{X}}{\overset{1}{A}} + \mathcal{L}_{\overset{4}{\xi}} \underset{\mathcal{X}}{\overset{0}{A}} + 3\mathcal{L}_{\overset{2}{\xi}}^2 \underset{\mathcal{X}}{\overset{0}{A}} + 4\mathcal{L}_{\overset{1}{\xi}} \mathcal{L}_{\overset{3}{\xi}} \underset{\mathcal{X}}{\overset{0}{A}} + 6\mathcal{L}_{\overset{1}{\xi}}^2 \mathcal{L}_{\overset{2}{\xi}} \underset{\mathcal{X}}{\overset{0}{A}} + \mathcal{L}_{\overset{1}{\xi}}^4 \underset{\mathcal{X}}{\overset{0}{A}}\,. \tag{4.3e}$$

Observe that for each term on the right hand side the perturbation order, given by the sum of the perturbation orders of $\underset{\mathcal{X}}{\overset{k}{A}}$ and the vector fields $\overset{k}{\xi}$ in the Lie derivatives, agrees with the perturbation order of the left hand side.

5 Gauge-invariant quantities

The main idea of gauge-invariant perturbation theory is to divide the variables describing the tensor field $\underset{\mathcal{X}}{A}_\epsilon$ into gauge independent variables $\mathbf{A}_\epsilon$, which describe properties of the physical field A_ϵ and hence observable quantities, and gauge dependent variables, which describe properties of the gauge only [Nak05, Nak07, Nak06]. One possibility to achieve this separation and to introduce gauge invariant quantities is to choose a distinguished gauge $\mathcal{S}_\epsilon$. This gauge can be obtained, for example, by imposing gauge conditions on the field $\underset{\mathcal{S}}{A}_\epsilon$; in the case of a metric, these could be the standard post-Newtonian or harmonic gauges. If these uniquely fix the gauge $\mathcal{S}_\epsilon$, we may use it to define the gauge invariant field as

$$\mathbf{A}_\epsilon = \underset{\mathcal{S}}{A}_\epsilon = \mathcal{S}_\epsilon^* A_\epsilon \,. \tag{5.1}$$

Given any other gauge $\mathcal{X}_\epsilon$, we can write the tensor field in this gauge in the form $\underset{\mathcal{X}}{A}_\epsilon = \mathcal{X}_\epsilon^*(\mathcal{S}_\epsilon^{-1})^*\mathbf{A}_\epsilon$, i.e., by applying a gauge transformation. We have thus achieved a split of $\underset{\mathcal{X}}{A}_\epsilon$ into a gauge dependent part, namely the diffeomorphism $\mathcal{S}_\epsilon^{-1} \circ \mathcal{X}_\epsilon$, describing the gauge, and a gauge-invariant part $\mathbf{A}_\epsilon$, describing the physical field. Note that this also implies a split of the number of free components of $\underset{\mathcal{X}}{A}_\epsilon$: the gauge invariant field $\mathbf{A}_\epsilon$ has fewer free components than the field in an arbitrary gauge, since some components are fixed by the gauge conditions corresponding to the choice of the distinguished gauge $\mathcal{S}_\epsilon$. These missing components are exactly found in the gauge transformations, if a different gauge $\mathcal{X}_\epsilon$ is chosen.
As discussed above, for any gauge transformation there exist vector fields, which we will now denote by $\overset{k}{X}$, such that the transformed perturbative expansion of a tensor field can be written as

$$\underset{\mathcal{X}}{\overset{k}{A}} = \sum_{0 \leq l_1 + 2l_2 + \ldots \leq k} \frac{k!}{(k - l_1 - 2l_2 - \ldots)!(1!)^{l_1}(2!)^{l_2} \cdots l_1! l_2! \cdots} \pounds^{l_1}_{\overset{1}{X}} \cdots \pounds^{l_j}_{\overset{j}{X}} \cdots \overset{k-l_1-2l_2-\ldots}{\mathbf{A}} \,. \tag{5.2}$$

The Taylor coefficients $\underset{\mathcal{X}}{\overset{k}{A}}$ thus also split into a gauge dependent part $\overset{k}{X}$ and a gauge-invariant part $\overset{k}{\mathbf{A}}$. Given any other gauge $\mathcal{Y}_\epsilon$, the same formula holds for a different family $\overset{k}{Y}$ of vector fields, but with the same gauge invariant part $\overset{k}{\mathbf{A}}$. The gauge defining vector fields $\overset{k}{X}$ and $\overset{k}{Y}$ are thus the only components of this split which change under a gauge transformation $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$. Writing the generating vector fields of Φ_ϵ as $\overset{k}{\xi}$, one finds that the transformation is given by

$$\overset{1}{Y} = \overset{1}{X} + \overset{1}{\xi} \,, \tag{5.3a}$$

$$\overset{2}{Y} = \overset{2}{X} + \overset{2}{\xi} + [\overset{1}{\xi}, \overset{1}{X}] \,, \tag{5.3b}$$

$$\overset{3}{Y} = \overset{3}{X} + \overset{3}{\xi} + 3[\overset{2}{\xi}, \overset{1}{X}] - [\overset{1}{\xi}, [\overset{1}{\xi}, \overset{1}{X}]] + 2[[\overset{1}{\xi}, \overset{1}{X}], \overset{1}{X}] \,, \tag{5.3c}$$

$$\begin{aligned}\overset{4}{Y} = {} & \overset{4}{X} + \overset{4}{\xi} + 3[\overset{2}{\xi}, \overset{2}{X}] + 4[\overset{3}{\xi}, \overset{1}{X}] + 6[[\overset{2}{\xi}, \overset{1}{X}], \overset{1}{X}] + 3[[\overset{1}{\xi}, \overset{1}{X}], \overset{2}{X}] - 3[\overset{2}{\xi}, [\overset{1}{\xi}, \overset{1}{X}]] \\ & + [\overset{1}{\xi}, [\overset{1}{\xi}, [\overset{1}{\xi}, \overset{1}{X}]]] + 3[[[\overset{1}{\xi}, \overset{1}{X}], \overset{1}{X}], \overset{1}{X}] - 3[[\overset{1}{\xi}, [\overset{1}{\xi}, \overset{1}{X}]], \overset{1}{X}]\end{aligned} \tag{5.3d}$$

and similarly for higher orders. The particular form of the gauge-invariant perturbations, of course, depends on the choice of the standard gauge $\mathcal{S}_\epsilon$, which must be adapted to the particular problem under consideration.

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